

Optimal Control with Fuzzy State Space Modeling Using Riccati Equation

Venkata Ramu G., Padmanabhan K., and Ananthi S.

Abstract—Fuzzy logic has a boon for nonlinear control systems. Normal fuzzy logic control with a proportional integral – Derivative (PID) controller is common. Control systems can be defined through transfer functions and state-space relations for linear systems. Optimal control to meet a performance index is possible only through State Space analysis. Optimal control in state space is centered around the Riccati Equation with state variable functions that has to be solved to yield the control law or trajectory. In the control scheme of an ozone generator, optimal control with a performance index had to be implemented. The method for finding the control functions by solving the equation graphically is described. The data is used for realizing an embedded control scheme for the generator

Index Terms—Fuzzy control, neuro-fuzzy systems, fuzzy system model, process control

I. INTRODUCTION

It is not often possible to develop a totally linear model of an interacting multivariable system so as to represent the same in linear state space formulation. The advent of fuzzy logic in control system modeling has proved useful [1] for characterizing nonlinear components, processes and also for identification of nonlinear systems [2]. We can easily specify a nonlinear surface by means of fuzzy logic based membership functions of the independent variables. For this purpose, two models - one by Mamdani [3] and another by Takagi and Sugeno [4] are available. Fuzzy State Space Model (FSSM) is a new modeling technique, which was developed for solving inverse problems in multivariable control systems. In this approach, the flexibility of fuzzy modeling is incorporated with the crisp state space models [5] proposed in the modern control theory. In [6], predictive functional control is combined with a fuzzy model of the process and formulated in the state space domain.

The prediction is based on a global linear model in the state space domain. In a recent paper, Adaptive fuzzy control of a class of MIMO nonlinear systems has been described by S.Labiod et al in [7]. Design of wide range optimal controller is described by Dragon Kukoli et al in [8].

These are but some of the attempts for implementation. To cite an example, in a distillation column, the feed preheating and bottom product composition controls are interacting and the equations are also nonlinear. While controlling the two variables of feed temperature and bottom product composition, the best or optimal control law that

minimizes input heat power is a requirement. The performance index is defined on the heat transfer relations and the best state space control is that which should minimize the heat power.

In state space methods, the control law function can be tailored to meet an optimized performance index. Theory relating to the technique of optimization is well described in text books [9]. The method is based on the solution of the Riccati equation. This is for linear systems. We need to optimize the control schemes more for nonlinear than for linear systems.

Several chemical engineering systems show nonlinear surfaces for output to input variable relations. It is these systems which need an optimization method to be employed in their control schemes so as to meet performance optimization such as power or reactant material. But these systems cannot be ordinarily written in terms of state space equations as in the linear types of systems. In our design of Ozone Generators [10] for water application, providing a controlled ozone concentration was the need. The reactor comprises of the ozone generating tube with high tension electric field applied. Air is inlet after drying and gets ozonised at exit of the tube. The tube is cooled by water to reduce temperature excess which decreases ozone production.

The air flow also affects ozone and is ordinarily kept constant from a pump. Control of high tension voltage is through pulse width modulation of the drive to the high voltage. Transformer. The relation between high tension and ozone produced is a nonlinear curve (Fig.1) in steady state and is also associated with a dead time constant. The power drawn increases beyond a certain high voltage applied.

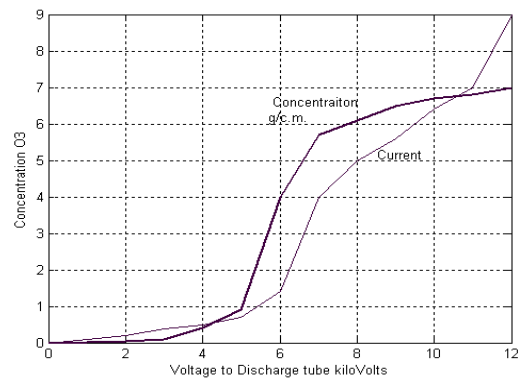


Fig. 1. The nonlinear relations of ozone production with high voltage

There is also another nonlinear relation between temperature and ozone, as well as voltage and current drawn. Thus the model for nonlinear ozone reactor control needed a fuzzy modeling in state space, because, state space methods for optimal control for a minimal power (current)

consumption is required. We developed fuzzy relations for ozone concentration variable and temperature variable by adjusting the membership functions to fit the nonlinear curves of experimental data [11] using Sugeno Fuzzy logic model.

With a PID controller we could control the ozone production. For large ozone generating systems with multiple reactor tubes, however, optimization of a performance index is essential. In [12], the discussions relating to power consumption is indicative of the need for optimal control.

In linear systems, the solution of a standard Riccati Equation is the method useful for obtaining the control relationship for optimal control. We use such an equation with a modification and apply it for fuzzy logic defined for a nonlinear state space system model of the ozone generator.

Section 2 reviews the method presently well known for linear system optimal control. The next section describes how our ozone generator model is placed in fuzzy state space. This is the reverse problem of finding the fuzzy membership functions and Sugeno relations to fit fig.1 and other nonlinear parameters. Having done this, we define the performance requirement (minimal power) and solve the Riccati equation in a novel way in the final section. That solution once evaluated is portable to an embedded microcontroller which is part of the ozone generator unit.

II. LINEAR STATE SPACE CONTROL SYSTEMS

With the usual notation, the state space model of an n^{th} order linear system with m inputs and l outputs is written as

$$\dot{x}' = A x + B u \quad (1)$$

$$y = C x + D u \quad (2)$$

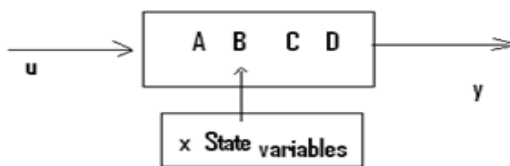


Fig. 2. Shows a State Space model

Before considering the fuzzy state feedback, let us consider a linear system with state variables and a normal controller for this system. The schematic of a full-state feedback system is given in fig.3.

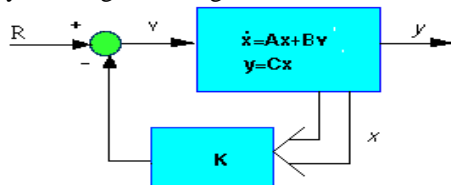


Fig. 3. State variable feedback with K.

With feedback vector K , the state equations are

$$\dot{x}' = (A - BK) x + Bv \quad (3)$$

$$y = C x \quad (4)$$

The characteristic equation is

$$\text{Det} \{sI - (A - BK)\} = 0 \quad (5)$$

The characteristic polynomial for this closed-loop system is the determinant of $(sI - (A - BK))$. Since A and $B \cdot K$ are

both matrices of $n \times n$, there will be n poles for the system. By using state feedback we have to modify the positions of the poles in the complex plane to be on the left half plane. Choice of pole positions will mean control law variations. If pole positions are chosen too far from the imaginary axis, the response will be very fast. But that would involve more power input to the system as well. Thus, a choice that meets a performance requirement and limitations of power or concentration of gas phase being controlled is to be implemented.

A. Principles of State Space Optimal Control and Riccati Equation

A performance index is represented in a quadratic form. For the case with just two penalty functions it is given by (using $Kx = u$)

$$\begin{aligned} J &= \int_0^{\infty} (x' Q x + u' R u) dt \\ &= \int_0^{\infty} (x' Q x + x' K' R K x) dt \\ &= \int_0^{\infty} x' (Q + K' R K) x dt \end{aligned} \quad (6)$$

where Q is the $(n \times n)$ state penalty matrix, R here is the $(m \times m)$ control penalty matrix.

J is a scalar quantity which is the total integrated quantity of penalty or power that the feedback control law $u = Kx$ will need and hence is a value that is best minimized under all control paths through the system variables every now and then.

The state vector x is multiplied by Q in $x' Q x$ and this represents something like the power input based on the states. For instance, in an electrical heating system, if I represents the current, the power is given by $I' R I = I^2 R$, the ohmic power. This is known as a Quadratic form.

Likewise, for the control vector u also, the quadratic form $u' R u$ indicates control power, which is a penalty paid for controlling the system. The matrix R will not contain any off-diagonal terms if each control variable decides its own penalty and is without any interaction with another variable.

Let us set

$$x' (Q + K' R K) x = d/dt (x' P x) \quad (7)$$

By which we assume that there exists a function matrix P , and a quadratic form $x' P x$ whose derivative is equal to the above integrand. By this assumption, we understand that the integral can be integrated to get a function P . This will enable the integral to be evaluated simply as $(x' P x)$.

Then,

$$\begin{aligned} x' (Q + K' R K) x &= -x' P x - x' P x' \\ &\text{(by derivative product rule)} \\ &= (A x - B K x)' P x - x' P (A x - B K x) \\ &= -x' \{ (A - B K)' P + P (A - B K) \} x \end{aligned} \quad (8)$$

Since this equation holds for all x , we can write from this:

$$-(Q + K' R K) = (A - BK)' P + P (A - BK)$$

$$PA + A' P + Q = -K' R K + (BK)' P - PBK$$

which can be simplified into

$$P B R^{-1} B' P = PA + A' P + Q \quad (10)$$

If $(A - BK)^t$ is a stable matrix, then by the Lyapunov equation, there exists a P matrix (positive definite) that satisfies the above equation.

After finding P by solving this, we can evaluate J . This is simple as $[x^t P x]_0$. Then, the control law is

$$u(t) \doteq -R^{-1} B^t P(t) \tag{11}$$

The steady state solution P_{SS} need not be unique. However, only one real positive definite solution can exist. Since P is of dimensions $(n \times n)$, we obtain n^2 equations to solve for the n^2 components of P . Yet due to the fact that it is symmetric, the number of computations is greatly reduced. The Matlab function *lqr* and *lqry* provide these utilities for linear systems for which the state space matrices are numerically defined..

III. MODELING OF NONLINEAR FUNCTIONS THROUGH FUZZY LOGIC IN STATE SPACE

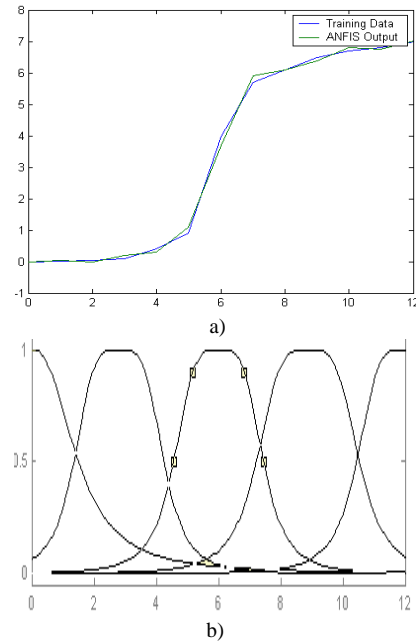
With nonlinear systems, in (1), the matrix A , B are not constants and vary with the vector x in state. For different ranges of x , one could prescribe a different A and B , though this is really not a tangible method. Instead, we can provide fuzzy sets for variables with defined membership functions and rules. Such attempts can be found in [13] which describes a frame work for Fuzzy Quantification in model analysis while [14] shows a formal approach to fuzzy modeling and systems.

It is shown in what follows that the method of characterizing any nonlinear function as in fig.1 can be done using ANFIS (Adaptive Neuro Fuzzy Inference System) for a Sugeno Model with input variable alone defined by membership functions and output is defined by linear equations with constant coefficients one for each rule in the set.

To characterize a nonlinear surface rather than a curve, two input variables and one output variable have to be defined with fuzzy logic along with a rule base for the combination. The details are given in reference [13] for an Ozone (Chemical) reactor process. There will be two nonlinear surfaces that need to be defined with appropriately adjusted fuzzy membership functions. What we describe further in this paper as an illustration for our fuzzy optimal control method can be used on such a model.

By choice of membership function, shape as well as its range, the nonlinearity relation can be realized. First by adjusting the ranges of the three member functions for each input and their shape, approximate fitting is made to match the curve as in fig.1. Then, the neural fuzzy inference training system (given in Matlab) is used to obtain the coefficients of the output equations for rules. (The functions used in Matlab are *genfis* and *anfis*. (Appendix 1).

An example illustrated in figure 4, shows how the relation of high tension supply versus ozone generation is a nonlinear relation which can be represented by five fuzzy membership functions and rules. Like this, we also modeled the ozone production versus reactor temperature and flow rate of gas. The state matrices A, B in state space have now become functions of fuzzy logic.



a) Five Bell shaped member functions.
b) The nonlinear function is modeled.

Fig. 4. Showing sugeno model based fitting of ozone nonlinear function for concentration and voltage to reactor. The output equations are given in Appendix.1.

IV. FUZZY OPTIMAL CONTROL – THE MODIFIED RICCATI EQUATION

In order to evolve a fuzzy relation based on the above optimal control for linear systems, let us trace the development of the nonlinear state space relations.

In fuzzy modeling, the function Ax is $f_a(x)$ and Bu becomes $f_b(u)$ or compositely as $f(x,u)$.

$$dx/dt=f(x,u) \tag{12}$$

These are functions with fuzzy memberships over various x . Now, as in (7),

$$d/dt (x^t P x) = -f(x,u)^t P x - x^t P f(x,u) \tag{13}$$

which we set equal to the performance index term (omitting Q) in (6) and keeping only the important control performance index term R which alone is related to power consumption due to control u . So by (7), we can write the above as equal to

$$= x^t K^t R K x = u^t R u \tag{14}$$

The function $f(x,u)$ is like the fuzzy function of fig.1 shown in fig.4. It can also be a graphically realized empirical function

A. Example illustrating the Solution of the Fuzzy logic Riccati Equation

For simplicity of illustration, let us assume two state variables and control variables (Concentration and voltage versus flow and temperature). Then, the function $f(x,u)$ is of the form

$$\begin{bmatrix} f1 \\ f2 \end{bmatrix}$$

Then $f^T(x, u)$ will be $| f1f2 |$

If P is then taken as in the above,

$$\begin{matrix} P11 & P12 \\ P21 & P22 \end{matrix} \text{ and } \begin{matrix} x1 \\ x2 \end{matrix} \quad (15)$$

So equation (13) will become on its right hand side,

$$\begin{matrix} | f1f2 | \\ P11 & P12 \\ P21 & P22 \end{matrix} \begin{matrix} x1 \\ x2 \end{matrix}$$

$$\begin{matrix} x1 & x2 \\ P11 & P12 \\ P21 & P22 \end{matrix} \begin{matrix} | f1 \\ | f2 \end{matrix} \quad (16)$$

Now if R , the performance matrix is given as:

$$\begin{matrix} | R11 & R12 \\ | R21 & R22 \end{matrix}$$

Then we get an equation involving functions, as:

$$\begin{matrix} | f1f2 | \\ P11 & P12 \\ P21 & P22 \end{matrix} \begin{matrix} x1 \\ x2 \end{matrix}$$

$$\begin{matrix} | x1 & x2 \\ P11 & P12 \\ P21 & P22 \end{matrix} \begin{matrix} | f1 \\ | f2 \end{matrix} \quad (17)$$

$$= x^t K^t R K x =$$

$$\begin{matrix} | x1 & x2 \\ K11 & K12 \\ K21 & K22 \end{matrix} \begin{matrix} | R11 & R12 \\ | R21 & R22 \end{matrix} \begin{matrix} | K11 & K21 \\ | K12 & K22 \end{matrix}$$

$$\begin{matrix} x1 \\ x2 \end{matrix} = \begin{matrix} u1 & u2 \\ R11 & R12 \\ R21 & R22 \end{matrix} \begin{matrix} u1 \\ u2 \end{matrix} \quad (18)$$

This right hand side is simply $u1^2 + u2^2$ if only R is an identity matrix. (Penalty terms equal for both control variables). Expanding the terms, we get

$$\begin{matrix} | f1f2 | \\ P11x1 & P12x2 \\ P21x1 & P22x2 \end{matrix}$$

$$\begin{matrix} | x1 & x2 \\ P11f1 & P12f2 \\ P21f1 & P22f2 \end{matrix} \quad (19)$$

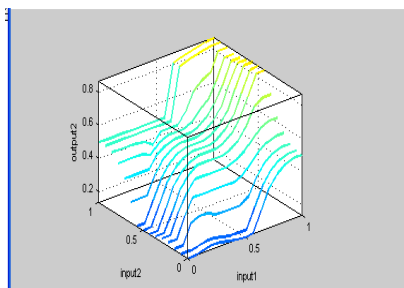


Fig. 5a

The above is the functional expression which has to be minimized over all possible values of the x trajectory during control. A function set for $f1, f2$ is as shown in fig5. There are two function surfaces, one for $x1'$ and another for $x2'$.

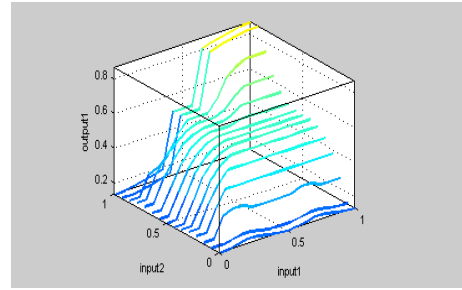


Fig. 5b

Fig. 5a Showing Contour plot of controlled variable -1 surface function $f1(x, u)$ for two valued inputs: Inputs 1 and 2. These function surfaces are developed using Sugeno Fuzzy logic fitting of experimental variable data over the ranges of inputs.

Fig.5b. Contour plot showing output variable 2.

The functions f is given for a model of the system, which give the contour surfaces. Let us for illustration; first choose a set of P values normalized to unity say as

$$P = [1 \ 0.8 ; 0.8 \ 1] \quad (20)$$

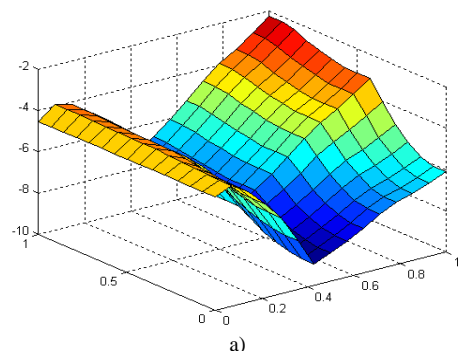
And evaluate the above expression for the entire range of x values using the above f function values shown. When, among these a minimum region of x and u are noticeable. For this region, the P values in (20) are the ones to be used in evaluating the expression and thereby finding the square sum of the control values (the right hand sided of 18).. Thus for this limited region of x , for this P matrix, the control values are found from P by (11):

$$u(t) \doteq -R^{-1}B'P(t) \quad (21)$$

The solution for the P which gives the least value near to zero for the equation (19) can be done by a Nelder Mead Search program such as given by the Matlab command *fminsearch*. The result is a P that is the optimal solution of the Riccati equation. The values for any set of R function can be tabulated.

The embedded microcontroller would hold the tabulated values in its nonvolatile memory.

In this work, the main theme is to illustrate the procedure to be employed for optimal control in a fuzzy logic modeled nonlinear multivariable control system. As far as fuzzy logical characterization of the system is concerned, the method employed is the ANFIS technique.. This technique is for Sugeno modeling of the nonlinear functions in state space. There are of course several techniques for fuzzy modeling some of which are to be found in references [15], [16].



a)

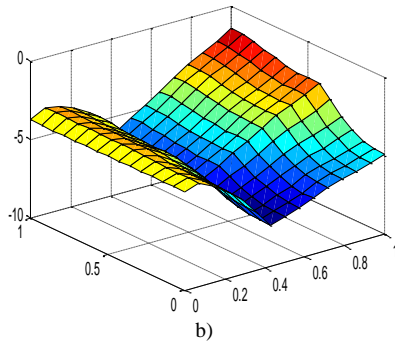


Fig. 7a. Showing the region of control vector space for one set of values of p_{11} , p_{12} , p_{21} , p_{22} . [$P = [1 \ 0.8 \ 0.8 \ 1]$] The input range [0.4-0.6 0.2-0.4] has this P as near optimal.

Fig. 7.b. Showing the region of control vector space for different values of p_{11} , p_{12} , p_{21} , p_{22} values ($P [1 \ 0.6 \ 0.6 \ 1]$) and $P [1 \ 0.9 \ 0.9 \ 1]$. The range [0.6-0.8 0.3-0.5] has this optimal P.

V. CONCLUSION

After an introduction to the use of state space optimal control in a linear system, the method useful for similar optimal control in fuzzy logic modeled nonlinear MIMO (multi input multi output) system is described. Thus the method of optimal control is easy by state space (SS) method with the Riccati equation solution for the variables controlled. Since this equation is a functional equation, it can be solved numerically. The same yields one new method for control signals to be applied over the entire range of state variables in this state space model with optimization of performance. Though the illustration taken was a simple two input two control variable model for graphical presentation of the principles, applications of the method can be used for many chemical process plant with multi variables and higher order equations.

APPENDIX I

This program when run will fit the fig.1 curve of concentration (not power) to a fuzzy model with member functions of the gbell type and equations for output.

```
v = (0:1:12)'; % V is voltage in kV
% c denotes concentration of ozone as in fig.1.
c=[0 0.02 0.04 0.1 .4 9 4 5.7 6.1 6.5 6.7 6.8 7]
epoch_n = 20;
in_fis = genfis1([v c'],5,'gbellmf'); % 5
memberships are chosen.
out_fis = anfis([v c'],in_fis,epoch_n);
plot(v,c,v,evalfis(v,out_fis));
legend('concentration Data','fuzzy modeled Output');
```

Then we type fuzzy at Command prompt. Then, we get the GUI. There, we import the two variable functions *in_fis* and *out_fis*. We look up at output functions and tabulate each of the five function coefficients.

0.141	0.082	0	0	0
0.385	0.728	0	0	0
0	2.82	13.1	0	0
0	0	0.79	0.96	0
0	0	0	0.498	0.878

APPENDIX II

The following listing of the Matlab program realizes for optimizing the performance index. for minimal optimal control vector determination in fuzzy **Riccati equation** solution: The fuzzy model of the system with two inputs and outputs is first defined through Matlab GUI "fuzzy" in the usual way as the variable saved as "vari" for use in this optimization program. Choices of P matrix are to be entered one after another and results of minimal points noted and saved for control in the entire region of state space. This can be saved as a function and used by *fminsearch* command to find the best value for P, starting with the value of P got be above trials.

%FuzRicati.m

```
fa=readfis('vari') % This is the 2 input 2 output fuzzy
inference system model for control
inputs=[1 2]
output=2
[x y z]=gensurf(fa,inputs,output)
% Finds the f2 function surface co-ordinates
output=1
[x1 y1 z1]=gensurf(fa,inputs,output)
% Finds the f1 function surface co-ordinates
output=1
z1(10,10) ; the value at input (10,10) for f1
z(10,10) ; the value at input(10,10) for f2
x1 ;
p11=1 ; here enter the P-matrix of choice
p12=.8
p21=0.8
p22=1
p=[p11 p12;p21 p22];
q1=z1*x1*1
q2=z*p12*x ; finds the values of the terms of the
equation given in (20)
r1=x1*p11*z1
r1=x1*p11*z1+x*p22*z1
r2=x1*p12*z+x*p22*z
q1
q2
r1
r2
% The final matrix is q1-r1 q2-r2 which gives the two
rows in equation (19)
a1=q1-r1
subplot (211)
surf(x,y,a1)
subplot(212)
a2 = q2-r2
surf( x,y,a2)
end
```

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