TRACKING CONTROL FOR NONLINEAR FLAT SYSTEMS BY LINEAR DYNAMIC OUTPUT FEEDBACK

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Abstract: In this contribution linear dynamic output feedback is used to achieve stable tracking for nonlinear flat systems. To this end a time varying differential operator representation resulting from a linearization of the plant about the reference trajectory is used. The control scheme is illustrated for a uniaxial vehicle model.

Keywords: flat systems, tracking control, polynomial approach, dynamic output feedback

1. INTRODUCTION

Since its introduction differential flatness (see e.g. (Fliess *et al.*, 1992) and (Martin *et al.*, 1997)) has proved as an important tool in nonlinear control theory. Many industrial applications have already been addressed. The flatness property of a dynamic system can be roughly characterized by the existence of a (fictitious) flat output, which parameterizes the input and the system states with a finite number of its time derivatives. Consequently, when a trajectory is planned for the flat output, the corresponding input can be obtained directly from the flat parameterization without computing solutions of differential equations. To stabilize the tracking of the trajectory, flat feedback is used. In general full state information is needed to realize the flat feedback. As a consequence, a nonlinear observer has to be designed. The time varying observer gain of this observer is determined on the basis of a linearization of the observer error dynamics about the reference trajectory such that methods for linear time varying systems can be used to stabilize the error dynamics (Fliess and Rudolph, 1996). The resulting dynamic output feedback controller is therefore only valid in the vicinity of the reference trajectory. The control scheme which is presented in the following uses the nonlinear feedforward controller resulting from the flatness property. The feedback part, however, is designed fully linear, using a linear differential operator representation of the plant. To this end, a linearization of the nonlinear system about the reference trajectory is used to get a linear time varying system description. The tracking is then stabilized via linear dynamic output feedback. This approach has already been proposed in (Deutscher, 2002). However, the obtained results are limited to systems where a linearization of the differential parameterization of the inputs and outputs yields a differential operator representation with column reduced denominator matrix. In Section 2 a new scheme for deriving the differential operator representation for static feedback linearizable systems is presented, guaranteeing the denominator matrix to be column reduced about the reference trajectory. Hence, the resulting differential operator representation is well defined for the complete trajectory. In Section 3 the design of dynamic output feedback controllers is presented which amounts to solving a time varying Diophantine equation. In Section 5 the introduced control scheme is applied to a uniaxial vehicle model. This example also demonstrates that the control scheme can easily be extended to dynamic feedback linearizable systems.

2. DIFFERENTIAL OPERATOR REPRESENTATION OF NONLINEAR FLAT SYSTEMS

2.1 Transformation to nonlinear controller form

Consider the following nth order nonlinear flat system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

$$y = h(x) \tag{2}$$

with m inputs u and m outputs y. If system (1)–(2) is static feedback linearizable (Isidori, 1995), then the definition of flatness simplifies to the following form: there exists a flat output

$$y_f = \Phi(x) \tag{3}$$

with $\dim(y_f) = \dim(u)$ such that

$$x = \psi_x(y_f, \dot{y}_f, ..., y_f^{(\beta)}) \tag{4}$$

$$u = \psi_u(y_f, \dot{y}_f, ..., y_f^{(\beta+1)}) \tag{5}$$

By setting $\beta + 1 = (\kappa_1, \ldots, \kappa_m)$ and introducing the (Brunovsky) states (Delaleau and Rudolph, 1998)

$$\begin{aligned} z &= [z_1^1, \ldots, z_{\kappa_1}^1, \ldots, z_1^m, \ldots, z_{\kappa_m}^m]^T \\ &= [y_{f1}, \ldots, y_{f1}^{(\kappa_1-1)}, \ldots \\ &\qquad \ldots, y_{fm}, \ldots, y_{fm}^{(\kappa_m-1)}]^T \end{aligned} \tag{6}$$

system (1) can be transformed via the diffeomorphism

$$x = \psi_x(z) \tag{7}$$

into the controller form (Isidori, 1995)

$$\dot{z}_k^i = z_{k+1}^i, \quad k = 1(1)\kappa_i - 1 \tag{8}$$

$$\dot{z}_{\kappa_i}^i = a^i(z) + \sum_{l=1}^{m} b_l^i(z) u_l, \quad i = 1(1)m \tag{9}$$

with the controllability indices $(\kappa_1, \ldots, \kappa_m)$ satisfying $\sum_{i=1}^m \kappa_i = n$. Whereby the diffeomorphism (7) results from inserting (6) in (4). In (8) and (9) the superscript i denotes the ith subsystem of the controller form. In order to express (2) in the z-coordinates, (4) is substituted in (2), yielding with (6)

$$y = h(\psi_x(y_f, \dot{y}_f, ..., y_f^{(\beta)})) = \bar{h}(z) \tag{10}$$

2.2 Derivation of the differential operator representation

In the following it is assumed that a reference trajectory $y_{f,d}(t)$ for $t \in [0, T]$ has been planned for the flat output (for further details see e.g. (Martin *et al.*, 1997)). The corresponding feedforward control u_d is then given by

$$u_d = \psi_u(y_{f,d}, \dot{y}_{f,d}, ..., y_{f,d}^{(\beta+1)}) \tag{11}$$

in view of (5) and the reference trajectory z_d in the z-coordinates results from inserting $y_{f,d}$ in (6). Furthermore, it is supposed in the sequel that the diffeomorphism (7) is well defined in a neighbourhood of z_d. As a consequence, the controller form (8)–(9) exists in a neighbourhood of z_d.

A linear time varying differential operator representation is computed using a linearization of system (8)–(9) about the reference trajectory z_d. To this end the relations (9) of all m subsystems of the controller form are written in the compact form

$$\dot{z}^* = a(z) + B(z)u \tag{12}$$

where

$$z^* = [z_{\kappa_1}^1, \ldots, z_{\kappa_m}^m]^T \tag{13}$$

$$a(z) = [a^1(z), \ldots, a^m(z)]^T \tag{14}$$

$$b^i(z) = [b_1^i(z), \ldots, b_m^i(z)]^T, \quad i = 1(1)m \tag{15}$$

$$B(z) = [b^1(z), \ldots, b^m(z)]^T \tag{16}$$

Linearizing (12) about (z_d, u_d) results in

$$\Delta\dot{z}^* = A(t)\Delta z + B(t)\Delta u \tag{17}$$

with $\Delta z^* = z^* - z_d^*$, $\Delta z = z - z_d$, $\Delta u = u - u_d$ and the matrices

$$A(t) = \left.\frac{\partial(a(z) + B(z)u)}{\partial z}\right|_{z_d, u_d} \tag{18}$$

$$B(t) = B(z_d(t)) \tag{19}$$

Using the differential operator $D = \frac{d}{dt}$, $\Delta\dot{z}^*$ can be expressed as

$$\Delta\dot{z}^* = \operatorname{diag}(D^{\kappa_i})\Delta y_f \tag{20}$$

where $\Delta y_f = y_f - y_{f,d}$. Introducing the blockdiagonal matrix

$$\begin{aligned} S(D) = \operatorname{diag}([1, \ldots, D^{\kappa_1 - 1}]^T, \ldots \\ \ldots, [1, \ldots, D^{\kappa_m - 1}]^T) \end{aligned} \tag{21}$$

it is furthermore possible to write

$$\Delta z = S(D)\Delta y_f \tag{22}$$

With (20) and (22) equation (17) reads

$$\operatorname{diag}(D^{\kappa_i})\Delta y_f = A(t)S(D)\Delta y_f + B(t)\Delta u \tag{23}$$

As the the controller form (9) is well defined in a neighbourhood of z_d by assumption, $B(z)$

(see (12)) is invertible in a neighbourhood of the trajectory z_d (Isidori, 1995). Consequently, $B(t)$ is nonsingular for $t \in [0,T]$ (see (19)) and thus (23) can be rearranged in the form

$$B^{-1}(t)\,(\mathrm{diag}(D^{\kappa_i}) - A(t)S(D))\triangle y_f = \triangle u \quad (24)$$

A linearization of (10) in the z-coordinates about z_d yields with $\triangle y = y - y_d$

$$\triangle y = \left.\frac{\partial \bar{h}(z)}{\partial z}\right|_{z_d} \triangle z = C(t)S(D)\triangle y_f \quad (25)$$

where (22) was used. By introducing the polynomial matrices $N(D,t)$ and $Z(D,t)$ according to

$$N(D,t) = B^{-1}(t)(\mathrm{diag}(D^{\kappa_i}) - A(t)S(D)) \quad (26)$$
$$Z(D,t) = C(t)S(D) \quad (27)$$

(see (24) and (25)) one finally obtains the linear time varying differential operator representation

$$N(D,t)\triangle y_f = \triangle u \quad (28)$$
$$\triangle y = Z(D,t)\triangle y_f \quad (29)$$

Note that (28) is a differential equation for $\triangle y_f$. The trajectory planning has to ensure that the elements of the matrices $A(t)$, $B^{-1}(t)$ and $C(t)$ are sufficiently smooth functions of time, such that by (26) and (27) the coefficients of $N(D,t)$ and $Z(D,t)$ are sufficiently smooth. With $\Gamma_c[N(D,t)]$ denoting the highest column degree matrix and $\delta_{ci}[N(D,t)]$ the ith column degree (Wolovich, 1974), $N(D,t)$ in (26) can be expressed as

$$N(D,t) = \Gamma_c[N(D,t)]\,\mathrm{diag}(D^{\delta_{ci}[N(D,t)]}) + N_R(D,t) \quad (30)$$

with $N_R(D,t)$ a polynomial matrix of lower degrees. Comparing (30) with (26) yields the following relations

$$\Gamma_c[N(D,t)] = B^{-1}(t) \quad (31)$$
$$\delta_{ci}[N(D,t)] = \kappa_i,\ \ i = 1(1)m \quad (32)$$

where (31) and (32) are implied by the nonlinear controller form (8)–(9) and are satisfied for $t \in [0,T]$. This shows that $N(D,t)$ is column reduced (Wolovich, 1974), i.e. $\det\Gamma_c[N(D,t] \neq 0$ for $t \in [0,T]$.

If the system (1)–(2) is not static feedback linearizable, it is always possible to introduce controller states (based on the differential parameterization (4)–(5)) such that the resulting system is static feedback linearizable. Then the differential operator representation can be derived as described above (see the example in Section 4 for details).

3. DESIGN OF LINEAR TRACKING CONTROLLERS

3.1 Time varying flat feedback

If external disturbances or modelling errors affect the tracking behaviour, a tracking controller is needed to stabilize the tracking. This the more, when the feedforward u_d is applied to the system (1)–(2), the dynamics of the tracking error $\triangle y_f = y_f - y_{f,d}$ in a neighbourhood of the trajectory is gouverned by the differential equation

$$N(D,t)\triangle y_f = 0 \quad (33)$$

The error dynamics (33) are time varying and might be too slow or even unstable, depending on the system (1)–(2) and the trajectory $y_{f,d}$. In order to assign a desired time invariant dynamics for the tracking error consider an additional control action $\triangle u$ (see (28)) in the form

$$\triangle u = \Gamma_c[N(D,t)]\bar{u} + N_R(D,t)\triangle y_f \quad (34)$$

with new input $\bar{u}$. Applying (34) to the differential operator representation (28) yields the time invariant dynamics

$$\mathrm{diag}(D^{\kappa_i})\triangle y_f = \bar{u} \quad (35)$$

in view of (30). Via the new input $\bar{u}$ it is possible to assign the stable time invariant dynamics

$$\mathrm{diag}(\tilde{n}_i(D))\triangle y_f = 0 \quad (36)$$

with the Hurwitz polynomials

$$\begin{aligned}\tilde{n}_i(D) &= D^{\kappa_i} + \tilde{a}_{i,\kappa_i-1}D^{\kappa_i-1} + \\ &\quad \ldots + \tilde{a}_{i,1}D + \tilde{a}_{i,0} \\ &= D^{\kappa_i} + \tilde{n}_{i,R}(D),\ \ i = 1(1)m \end{aligned} \quad (37)$$

to the tracking error $\triangle y_f$, by setting

$$\bar{u} = -\,\mathrm{diag}(\tilde{n}_{i,R}(D))\triangle y_f \quad (38)$$

Substituting (38) in (34) yields the corresponding time varying flat feedback

$$\begin{aligned}\triangle u = &-\Gamma_c[N(D,t)]\,\mathrm{diag}(\tilde{n}_{i,R}(D))\triangle y_f \\ &+N_R(D,t)\triangle y_f\end{aligned} \quad (39)$$

Adding

$$\begin{aligned}&\Gamma_c[N(D,t)]\,\mathrm{diag}(D^{\kappa_i})\triangle y_f \ldots \\ &\quad \ldots - \Gamma_c[N(D,t)]\,\mathrm{diag}(D^{\kappa_i})\triangle y_f\end{aligned} \quad (40)$$

to (39) leads to the more compact formulation

$$\triangle u = (N(D,t) - \tilde{N}(D,t))\triangle y_f \quad (41)$$

with the polynomial matrix

$$\tilde{N}(D,t) = \Gamma_c[N(D,t)]\,\mathrm{diag}(\tilde{n}_i(D)) \quad (42)$$

The tracking controller design results in the overall control law

$$u = u_d + \triangle u \quad (43)$$

With this control strategy only the the feedforward u_d (see (11)) is applied when the system (1)–(2) exactly tracks the reference trajectory $y_{f,d}$ (i.e. $\triangle y_f = 0$).

3.2 Dynamic time varying output feedback

If the tracking error $\triangle y_f$ or its time derivatives are not available for measurement, the flat feedback (41) can at least be estimated by a time varying output feedback. In the following, a generally applicable approach to this problem is described resulting in a dynamic feedback controller of relatively high order. However, in some cases and for specific assignments for the dynamics of the tracking error $\triangle y_f$ (36) it may also be possible to find a reduced-order or even static implementation.

Consider the following estimate

$$\triangle\hat{u} = (N(D,t) - \tilde{N}(D,t))\triangle\hat{y}_f \tag{44}$$

of the control law (41) where $\triangle\hat{u}$ and $\triangle\hat{y}_f$ denote estimates for $\triangle u$ and $\triangle y_f$ respectively. For the dynamics of the estimation error $\triangle\hat{u} - \triangle u$ the following homogeneous differential equation can be assigned

$$\Delta(D)(\triangle\hat{u} - \triangle u) = 0 \tag{45}$$

If $\Delta(D)$ is a stable polynomial matrix (i.e. $\det \Delta(D)$ is a Hurwitz polynomial), the estimation error decays to zero. The dynamic output feedback controller realizing the error dynamics (45) is obtained by substituting (41) in the right hand side of

$$\Delta(D)\triangle\hat{u} = \Delta(D)\triangle u \tag{46}$$

yielding

$$\Delta(D)\triangle\hat{u} = \Delta(D)(N(D,t) - \tilde{N}(D,t))\triangle y_f \tag{47}$$

Since the dynamic output feedback controller uses $\triangle u$ and $\triangle y$ as inputs, one has to set

$$\begin{aligned}\Delta(D)\triangle\hat{u} &= \Delta(D)(N(D,t) - \tilde{N}(D,t))\triangle y_f \\ &\overset{!}{=} Z_u(D,t)\triangle u + Z_y(D,t)\triangle y \end{aligned} \tag{48}$$

Substituting $\triangle u = N(D,t)\triangle y_f$ (see (28)) and $\triangle y = Z(D,t)\triangle y_f$ (see (29)) in (48) yields

$$\begin{aligned}\Delta(D)\triangle u &= \Delta(D)(N(D,t) - \tilde{N}(D,t))\triangle y_f \\ &= Z_u(D,t)\triangle u + Z_y(D,t)\triangle y \\ &= (Z_u(D,t)N(D,t) + \ldots \\ &\quad \ldots Z_y(D,t)Z(D,t))\triangle y_f \end{aligned} \tag{49}$$

From (49) the Diophantine equation

$$\begin{aligned}Z_u(D,t)N(D,t) + Z_y(D,t)Z(D,t) = \\ \Delta(D)(N(D,t) - \tilde{N}(D,t))\end{aligned} \tag{50}$$

is deduced, which has to be solved in order to determine the unknown matrices $Z_u(D,t)$ and $Z_y(D,t)$. In (Limanond and Tsakalis, 2001) conditions for the existence of a solution of (50) are given, which can be used to find a general form of $Z_u(D,t)$ and $Z_y(D,t)$ consisting of polynomials in D with unknown time varying coefficients. These coefficients are determined by a generally under-determined system of linear time varying equations, which is easily derived by inserting the general expressions for $Z_u(D,t)$ and $Z_y(D,t)$ into (50) and reordering subsequently. The resulting set of equations is only solvable, if the Sylvester matrix related to the pair $Z(D,t)$ and $N(D,t)$ has full rank (see (Limanond and Tsakalis, 2001)). As this can always be assured by suitable trajectory planning, it does not mean serious restrictions. Moreover, if the system of equations is very complex, no general symbolic solution may be found. In any case, however, sample values for the elements of $Z_u(D,t)$ and $Z_y(D,t)$ can be calculated numerically for any discrete points in time of interest. Remaining degrees of freedom may be used to simplify the solution or to achieve additional requirements of the controller (see also the example in Section 4). Accordingly, the dynamic output feedback controller resulting from (48) and (43) can be summarized as follows

$$\Delta(D)\triangle\hat{u} = Z_u(D,t)\triangle u + Z_y(D,t)\triangle y \tag{51}$$

$$u = u_d + \triangle\hat{u} \tag{52}$$

If the initial conditions of the system (1)–(2) are not significantly inconsistent with the starting point $y_{f,d}(0)$ of the reference trajectory and no disturbances are present, the error $\triangle\hat{u} - \triangle u$ will start converging to zero independently from the inputs of the feedback controller $\triangle u$ and $\triangle y$. As a result, when assuming that the dynamics chosen in (45) is sufficiently fast, the controller (51)–(52) stabilizes the tracking according to the differential equation (36). Even if disturbances must be taken into account, this conclusion remains valid as long as the caused errors $\triangle\hat{u} - \triangle u$ are small.

4. EXAMPLE

In the following the normalized third order ($n = 3$) state space representation of a uniaxal vehicle

$$\begin{aligned}\dot{x}_1 &= \tfrac{1}{2}(u_1 + u_2)\cos x_3 \\ \dot{x}_2 &= \tfrac{1}{2}(u_1 + u_2)\sin x_3 \\ \dot{x}_3 &= \tfrac{1}{2}(u_1 - u_2)\end{aligned} \tag{53}$$

$$y = [y_1, y_2]^T = [x_1, x_2]^T \tag{54}$$

is considered, where the inputs u_1 and u_2 are the the left and right track velocity, the outputs y_1 and y_2 represent the position in a fixed Cartesian coordinate system and the state x_3 means the angle between vehicle velocity and y_1-axis. A flat output y_f is given by the physical output y (see (Martin *et al.*, 1997)), i.e.

$$y_f = [y_{f1}, y_{f2}]^T = y \tag{55}$$

As reference trajectory

$$y_{f,d} = [r \sin \tfrac{v_0}{r} t, -r \cos \tfrac{v_0}{r} t]^T \tag{56}$$

is assigned, which describes a circular movement with radius $r = 10$ at the constant speed $v_0 = 5$.

In contrast to the assumptions in Section 2.1, the model (53) is not static feedback linearizable. This can easily be seen by evaluating relationship (5) to

$$\begin{aligned} u &= \psi_u(y_{f1}, ..., y_{f1}^{(\kappa_1)}, y_{f2}, ..., y_{f2}^{(\kappa_2)}) \\ &= \psi_u(y_{f1}, \dot{y}_{f1}, \ddot{y}_{f1}, y_{f2}, \dot{y}_{f2}, \ddot{y}_{f2}) \end{aligned} \tag{57}$$

from which

$$\kappa_1 + \kappa_2 = 2 + 2 > n = 3 \tag{58}$$

can be deduced. However, the problem can be solved by introducing the controller state

$$x_4 = \sqrt{\dot{y}_{f1}^2 + \dot{y}_{f2}^2} \tag{59}$$

with $\frac{1}{2}(u_1 + u_2) = x_4$ and the new inputs

$$v = [v_1, v_2]^T = [\dot{x}_4, \tfrac{1}{2} u_2]^T \tag{60}$$

resulting in the extended static feedback linearizable system described by

$$\begin{aligned} \dot{x}_1 &= x_4 \cos x_3 \\ \dot{x}_2 &= x_4 \sin x_3 \\ \dot{x}_3 &= x_4 - 2v_2 \\ \dot{x}_4 &= v_1 \end{aligned} \tag{61}$$

$$\overline{y} = [y_1, y_2, x_4]^T \tag{62}$$

Note that the measurable controller state x_4 is included as additional output in (62). Analogously to (5) and (11), the new input v can be expressed in terms of the flat output y_f as

$$v = \psi_v(\dot{y}_{f1}, \ddot{y}_{f1}, \dot{y}_{f2}, \ddot{y}_{f2}) \tag{63}$$

and the feedforward controller for the extended system (61) is directly given as

$$v_d = \psi_v(\dot{y}_{f1,d}, \ddot{y}_{f1,d}, \dot{y}_{f2,d}, \ddot{y}_{f2,d}) \tag{64}$$

by inserting the reference trajectory (56) into (63). The extended state space representation (61) is converted into controller form by employing the state transformation

$$z = [y_{f1}, \dot{y}_{f1}, y_{f2}, \dot{y}_{f2}]^T \tag{65}$$

$$= [x_1, x_4 \cos x_3, x_2, x_4 \sin x_3]^T \tag{66}$$

(see (6)). Subsequently, the result (see (8)–(9)) is linearized about the reference trajectory z_d given by

$$z_d = [y_{f1,d}, \dot{y}_{f1,d}, y_{f2,d}, \dot{y}_{f2,d}]^T \tag{67}$$

and v_d (see (64)), which yields

$$\begin{aligned} \Delta \dot{z}^* &= \begin{bmatrix} 0 & -5\cos\frac{t}{2}\sin\frac{t}{2} & 0 & -\frac{11}{2}+5\cos^2\frac{t}{2} \\ 0 & \frac{1}{2}+5\cos^2\frac{t}{2} & 0 & 5\cos\frac{t}{2}\sin\frac{t}{2} \end{bmatrix} \Delta z \\ &+ \begin{bmatrix} \cos\frac{t}{2} & 10\sin\frac{t}{2} \\ \sin\frac{t}{2} & -10\cos\frac{t}{2} \end{bmatrix} \Delta v \end{aligned} \tag{68}$$

$$\Delta \overline{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \cos\frac{t}{2} & 0 & \sin\frac{t}{2} \end{bmatrix} \Delta z \tag{69}$$

(see (17)) where $\Delta z^* = [\Delta z_2, \Delta z_4]^T$ according to (13), $\Delta v = v - v_d$ and $\Delta \overline{y} = \overline{y} - \overline{y}_d$. Using these results the polynomial matrices

$$N(D,t) = \begin{bmatrix} \cos\frac{t}{2}D^2 - \frac{1}{2}\sin\frac{t}{2}D & \sin\frac{t}{2}D^2 + \frac{1}{2}\cos\frac{t}{2}D \\ \frac{1}{10}\sin\frac{t}{2}D^2 + \frac{11}{20}\cos\frac{t}{2}D & -\frac{1}{10}\cos\frac{t}{2}D^2 + \frac{11}{20}\sin\frac{t}{2}D \end{bmatrix} \tag{70}$$

and

$$Z(D,t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \cos\frac{t}{2}D & \sin\frac{t}{2}D \end{bmatrix} \tag{71}$$

determining the differential operator representation (28) and (29) can be calculated by means of (26) and (27). The dynamics of the tracking error $\Delta y_f = y_f - y_{fd}$ is specified by

$$\widetilde{N}(D,t) = \Gamma_c[N(D,t)]\, \mathrm{diag}(D^2 + 2D + 1, D^2 + 2D + 1) \tag{72}$$

(see 42) and the dynamics of the error $\Delta\hat{v} - \Delta v$ is assigned by

$$\Delta(D) = \begin{bmatrix} D+10 & 0 \\ 0 & D+10 \end{bmatrix} \tag{73}$$

Using the conditions established in (Limanond and Tsakalis, 2001) it can be shown that the Diophantine equation (50) can be fulfilled by polynomial matrices $Z_v(D,t)$ and $Z_{\overline{y}}(D,t)$ with $\deg(Z_v) = 0$ and $\deg(Z_{\overline{y}}) = 1$. Thus, the solution can be written in the general form

$$Z_v(D,t) = \begin{bmatrix} z_{v11}^0(t) & z_{v12}^0(t) \\ z_{v21}^0(t) & z_{v22}^0(t) \end{bmatrix} \tag{74}$$

$$Z_{\overline{y}}(D,t) = \begin{bmatrix} z_{\overline{y}11}^1(t)D + z_{\overline{y}11}^0(t) & z_{\overline{y}12}^1(t)D + z_{\overline{y}12}^0(t) & z_{\overline{y}13}^0(t) \\ z_{\overline{y}21}^1(t)D + z_{\overline{y}21}^0(t) & z_{\overline{y}22}^1(t)D + z_{\overline{y}22}^0(t) & z_{\overline{y}23}^0(t) \end{bmatrix} \tag{75}$$

Note that the first-order terms in the last column of $Z_{\overline{y}}(D,t)$ are omitted, since $\dot{\overline{y}}_3 = \dot{x}_4 = v_1$ (see (60)) meaning that without loss of generality these coefficients can be included in the zero-order terms in the first column of $Z_v(D,t)$. Inserting the general expressions (74)–(75) into (50) leads to a system of 12 independent linear time varying equations for the 14 unknown coefficients in $Z_v(D,t)$ and $Z_{\overline{y}}(D,t)$, which can be solved with the assistance of the computer algebra system MAPLE. The two resulting degrees of freedom in the solution are determined by the condition that all coefficients in $Z_v(D,t)$ and $Z_{\overline{y}}(D,t)$ must remain bounded for all times t. Hence, the dynamic output feedback controller (51) is specified completely. One possible realization of the controller (51) is obtained by introducing the controller states $\Delta\xi = [\Delta\xi_1, \Delta\xi_2]^T$ according to

$$\Delta\xi_1 = \Delta\hat{v}_1 - z^1_{\overline{y}11}(t)\Delta\overline{y}_1 - z^1_{\overline{y}12}(t)\Delta\overline{y}_2 \quad (76)$$

$$\Delta\xi_2 = \Delta\hat{v}_2 - z^1_{\overline{y}21}(t)\Delta\overline{y}_1 - z^1_{\overline{y}22}(t)\Delta\overline{y}_2 \quad (77)$$

which results in the state space representation

$$\Delta\dot{\xi} = \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix}\Delta\xi + \begin{bmatrix} -2 & -5 \\ \frac{11}{20} & -2 \end{bmatrix}\Delta v +$$
$$\begin{bmatrix} -\frac{15}{2}\cos\frac{t}{2} + \frac{101}{2}\sin\frac{t}{2} & -\frac{15}{2}\cos\frac{t}{2} - \frac{101}{2}\sin\frac{t}{2} & -\frac{37}{2} \\ 20\sin\frac{t}{2} + \cos\frac{t}{2} & -20\cos\frac{t}{2} + \sin\frac{t}{2} & \frac{13}{2} \end{bmatrix}\Delta\overline{y} \quad (78)$$

$$\Delta\hat{v} = \Delta\xi + \begin{bmatrix} -5\sin\frac{t}{2} & 5\cos\frac{t}{2} & 0 \\ -\frac{21}{10}\sin\frac{t}{2} & \frac{21}{10}\cos\frac{t}{2} & 0 \end{bmatrix}\Delta\overline{y} \quad (79)$$

Thus, stabilized tracking is achieved by the inputs (see (52))

$$v_1 = v_{1,d} + \Delta\hat{v}_1 \quad (80)$$

$$v_2 = v_{2,d} + \Delta\hat{v}_2 \quad (81)$$

where $v_{1,d}$ and $v_{2,d}$ are determined by the nonlinear feedforward controller (64). Note that the overall order of the so defined controller is three, since not only a realization of the state space model (78)–(79) must be provided but also an implementation of the system extension (59)–(60). The presented feedback design was simulated applying MATLAB/SIMULINK and the results were compared to the well-established flatness-based approach using a dynamic nonlinear state feedback controller combined with a nonlinear tracking observer with a linear time varying observer gain (see e.g. (Fliess and Rudolph, 1996) for details). Although the order of the considered nonlinear controller was four, i.e. by one higher than the corresponding order of the linear design, both offered a very similar performance assuming small errors in the initial state, which is shown in Figure 1 for the initial state errors $x_1(0) - x_{1,d}(0) = -0.05r$, $x_2(0) - x_{2,d}(0) = 0.05r$ and $x_3(0) - x_{3,d}(0) = \frac{\pi}{18}$. Moreover, even for relatively large errors and disturbances the simulated dynamic behavior did not worsen significantly when using the linear approach.

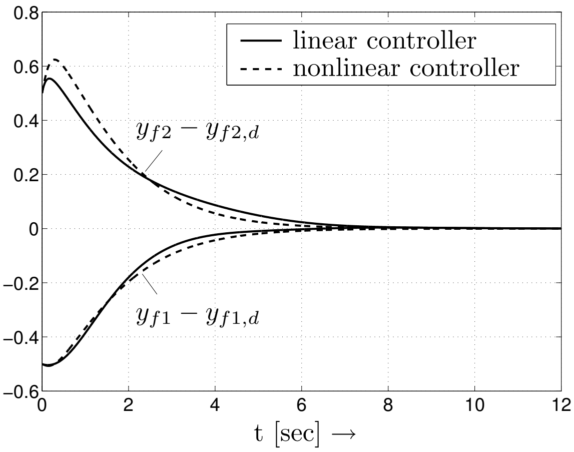

Fig. 1. Comparison of the tracking errors with linear and nonlinear tracking controller

5. CONCLUSIONS

In this contribution a systematic design procedure for linear dynamic output feedback controllers for nonlinear flat systems has been developed. The controller achieves time invariant tracking error dynamics in the vicinity of the reference trajectory. The applicability of the proposed scheme has been shown for a uniaxial vehicle model. Even more general linear time varying system descriptions than (28)–(29) could be used for the controller design, see e.g. (Levine and Nguyen, 2003).

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