Low-Complexity SSOR-Based Precoding for Massive MIMO Systems

Tian Xie, Linglong Dai, Xinyu Gao, Xiaoming Dai, and Youping Zhao

***Abstract*—With the increase of the number of base station (BS) antennas in massive multiple-input multiple-output (MIMO) systems, linear precoding schemes are able to achieve the near-optimal performance, and thus are more attractive than nonlinear precoding techniques. However, conventional linear precoding schemes such as zero-forcing (ZF) precoding involve the matrix inversion of large size with high computational complexity, especially in massive MIMO systems. To reduce the complexity, in this letter, we propose a low-complexity linear precoding scheme based on the symmetric successive over relaxation (SSOR) method. Moreover, we propose a simple way to approximate the optimal relaxation parameter of the SSOR-based precoding by exploiting the channel property of asymptotical orthogonality in massive MIMO systems. We show that the proposed SSOR-based precoding can reduce the complexity of the classical ZF precoding by about one order of magnitude without performance loss, and it also outperforms the recently proposed linear approximate precoding schemes in typical fading channels.**

***Index Terms*—Massive multiple-input multiple-output (MIMO), linear precoding, SSOR method.**

I. Introduction

UNLIKE conventional small-scale multiple-input multiple-output (MIMO), massive MIMO employs a large number of antennas (e.g., in the order of hundreds) at the base station (BS) to simultaneously serve a set of users (e.g., in the order of tens). Massive MIMO features great improvement in spectral and energy efficiency [1], and it is widely recognized as a very promising technology for the 5th generation (5G) wireless communication systems [2].

However, there are several challenges to be solved for practical massive MIMO systems, one of which is the low-complexity precoding scheme with near-optimal performance. In massive MIMO systems, linear precoding techniques with reduced complexity can also achieve the capacity-approaching performance due to the channel property of asymptotical orthogonality [2], which makes them more attractive for massive MIMO systems. Unfortunately, most linear precoding schemes such as zero-forcing (ZF) precoding require the matrix inversion of large size, which still results in high complexity when the dimension of MIMO channels grows large in massive MIMO systems. Recently, some efforts have been endeavored to further reduce the complexity of linear precoding schemes [3], [4]. Among them, a truncated polynomial expansion (TPE) precoding scheme is proposed in [4] to avoid the matrix inversion. However, TPE precoding has to deal with complicated parameter optimization problems. Another low-complexity linear precoding scheme without complicated parameters is Neumann-based precoding, which avoids matrix inversion by Neumann series [3]. Neumann-based precoding enjoys reduced complexity when the number of iterations is small, but it exhibits the same order of complexity as the classical ZF precoding when a large number of iterations is required to achieve the near-optimal performance.

In this letter, we propose a low-complexity linear precoding scheme based on the symmetric successive over relaxation (SSOR) method to substantially reduce the complexity of the classical ZF precoding for massive MIMO systems. Specifically, the proposed SSOR-based precoding exploits the channel property of asymptotical orthogonality in massive MIMO systems to reduce the complexity of linear precoding, which is realized by computing the expected precoded signal with the classical successive over relaxation (SOR) method in both the forward and reverse order. To ensure the performance of SSOR-based precoding, we also propose a simple way to determine the optimal relaxation parameter by utilizing the channel property of asymptotical orthogonality in massive MIMO systems, which only depends on the dimension of massive MIMO systems. The analysis shows that the proposed SSOR-based precoding can reduce the complexity by about one order of magnitude. In addition, simulation results show that the proposed SSOR-based precoding outperforms Neumann-based precoding and TPE-based precoding, and it can also achieve the near-optimal performance with a small number of iterations.

Notations: Lower-case and upper-case boldface letters denote vectors and matrices. $(\cdot)^H$, $(\cdot)^\dagger$, and $(\cdot)^{-1}$ present the Hermitian conjugation, pseudo-inverse, and inversion of a matrix, respectively. Finally, $\mathcal{CN}(a, \sigma^2\mathbf{I})$ denotes the circularly symmetric complex Gaussian distribution with mean a and covariance matrix $\sigma^2\mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

II. System Model

We consider the typical massive MIMO system, where the BS employs N antennas to simultaneously serve K single-antenna users, and we usually have N much larger than K, i.e., $N \gg K$ [2]. Denoting the transmitted signal vector after

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precoding by $\mathbf{x} \in \mathbb{C}^{N\times 1}$ in the downlink, we can express the received signal vector $\mathbf{y} \in \mathbb{C}^{K\times 1}$ for K users as

$$\mathbf{y} = \sqrt{\rho_f}\,\mathbf{Hx} + \mathbf{n}, \tag{1}$$

where ρ_f is the downlink transmit power, $\mathbf{H} \in \mathbb{C}^{K\times N}$ denotes the flat Rayleigh fading channel matrix whose entries follow the distribution $\mathcal{CN}(0, 1)$, and $\mathbf{n} \in \mathbb{C}^{K\times 1}$ denotes the additive white Gaussian noise vector, which follows the distribution $\mathcal{CN}(0, \sigma^2\mathbf{I})$. When the linear precoding scheme such as ZF precoding is used, we can express $\mathbf{x}$ as

$$\mathbf{x} = \mathbf{Gs}, \tag{2}$$

where $\mathbf{G} \in \mathbb{C}^{N\times K}$ denotes the linear precoding matrix, and $\mathbf{s} \in \mathbb{C}^{K\times 1}$ is the source signal vector for K different users.

III. Proposed SSOR-Based Precoding

A. Conventional ZF Precoding

We first briefly review the classical ZF precoding. According to [2], the conventional ZF precoding matrix $\mathbf{G}_{\text{ZF}}$ can be expressed as

$$\mathbf{G}_{\text{ZF}} = \beta_{\text{ZF}}\mathbf{H}^{\dagger} = \beta_{\text{ZF}}\mathbf{H}^H(\mathbf{HH}^H)^{-1} = \beta_{\text{ZF}}\mathbf{H}^H\mathbf{P}^{-1}, \tag{3}$$

where β_{ZF} denotes the power normalization factor that can be selected as $\beta_{\text{ZF}} = \sqrt{\frac{K}{\text{tr}(\mathbf{P}^{-1})}}$ [1], and $\mathbf{P} = \mathbf{HH}^H$. Based on (1), (2), and (3), the equivalent channel matrix can be presented as

$$\mathbf{W} = \mathbf{HG}_{\text{ZF}} = \beta_{\text{ZF}}\mathbf{HH}^H\mathbf{P}^{-1}. \tag{4}$$

As shown in (3), ZF precoding requires the matrix inversion of large size, so its complexity $\mathcal{O}(K^3)$[1] rises rapidly as the dimension of massive MIMO expands.

B. Proposed SSOR-Based Precoding

Considering (2) and (3), we can rewrite the transmitted signal vector $\mathbf{x}$ as

$$\mathbf{x} = \beta_{\text{ZF}}\mathbf{H}^H\mathbf{P}^{-1}\mathbf{s} = \beta_{\text{ZF}}\mathbf{H}^H\mathbf{t}, \tag{5}$$

where $\mathbf{t} = \mathbf{P}^{-1}\mathbf{s}$. Equivalently, we have

$$\mathbf{Pt} = \mathbf{s}. \tag{6}$$

The SSOR method can achieve the expected precoded vector $\mathbf{t}$ in (6) in an iterative way without the complicated matrix inversion $\mathbf{P}^{-1}$ of large size. The premise of utilizing SSOR method is that the matrix $\mathbf{P}$ should be Hermitian positive definite. Firstly, due to the fact that $\mathbf{P} = (\mathbf{HH}^H)$ according to the definition, the matrix $\mathbf{P}$ is clear to be Hermitian. Secondly, by denoting an arbitrary nonzero vector as $\mathbf{u}$, we have

$$\mathbf{uGu}^H = \mathbf{u}\left(\mathbf{HH}^H\right)\mathbf{u}^H = \mathbf{uHH}^H\mathbf{u}^H = \mathbf{uH}\,(\mathbf{uH})^H. \tag{7}$$

[1]Note that there are recursive matrix inversion algorithms of which the complexity is less than $\mathcal{O}(K^3)$ (e.g., the Strassen algorithm [5]). However, these algorithms are hard to be implemented in practice due to the complicated recursive structure. The practical matrix inversion algorithms such as Gauss-Jordan elimination method have the complexity of $\mathcal{O}(K^3)$.

In Rayleigh fading channels, where the channel matrix $\mathbf{H}$ is a full-rank matrix, $\mathbf{uH}$ equals to a zero vector only when $\mathbf{u}$ is a zero vector. So we have $\mathbf{uH}\,(\mathbf{uH})^H > 0$ for all nonzero vectors, which indicates that $\mathbf{P}$ is positive definite. Thus, we propose the SSOR-based precoding by utilizing SSOR method to solve (6) in the following three steps:

1) Decompose $\mathbf{P}$ as

$$\mathbf{P} = \mathbf{D} + \mathbf{L} + \mathbf{L}^H, \tag{8}$$

where $\mathbf{D}$, $\mathbf{L}$, and $\mathbf{L}^H$ denote the diagonal component, the strictly lower triangular component, and the strictly upper triangular component of $\mathbf{P}$, respectively. Here, we write the strictly upper triangular matrix as $\mathbf{L}^H$ since $\mathbf{P}$ is Hermitian positive definite as discussed above.

2) Take successive over relaxation (SOR) method in the forward order to compute the first half iteration by

$$(\mathbf{D} + \omega\mathbf{L})\mathbf{t}^{(i+1/2)} = (1-\omega)\mathbf{Dt}^{(i)} - \omega\mathbf{L}^H\mathbf{t}^{(i)} + \omega\mathbf{s}, \tag{9}$$

where the superscript i denotes the number of iterations, and ω is the relaxation parameter, which will be discussed in details later.

3) Take SOR method in the reverse order to compute the second half iteration of by

$$(\mathbf{D} + \omega\mathbf{L}^H)\mathbf{t}^{(i+1)} = (1-\omega)\mathbf{Dt}^{(i+1/2)} - \omega\mathbf{Lt}^{(i+1/2)} + \omega\mathbf{s}. \tag{10}$$

After several times of iterations based on (9) and (10), the obtained vector $\mathbf{t}$ can be multiplied by the matrix $\beta_{\text{ZF}}\mathbf{H}^H$ to achieve the signal vector $\mathbf{x}$ for transmission after precoding as shown in (5). Note that $\mathbf{t}$ needs to be recomputed for each source signal vector $\mathbf{s}$. Thus, the proposed SSOR-based precoding is more suitable for the fast time-varying massive MIMO systems (e.g., the communications between BS and users in fast moving trains or vehicles), where the coherence interval is small due to the high speed of users.

We can find that the proposed SSOR-based precoding solves the complicated matrix inversion required by ZF precoding through an iterative approach, which has much lower complexity as will be quantified in Section III-D. We have found other papers that used the similar iterative methods to reduce the complexity of signal precessing in massive MIMO systems [6], [7]. In [6], the authors proposed a low-complexity signal detection algorithm using Richardson method, and a SOR-based signal detection algorithm was proposed in [7]. However, our proposed SSOR-based precoding is different from these existing works. Compared with the Richardson-based signal detection [6], the proposed SSOR-based precoding exploits the diagonally dominant structure of the Gram matrix $\mathbf{P}$ in massive MIMO systems, so that we can only focus on the dominant diagonal elements of $\mathbf{P}$ while neglecting the unimportant non-diagonal elements of $\mathbf{P}$. Compared with the SOR-based signal detection algorithm [7], the proposed SSOR-based precoding is less sensitive to the relaxation parameter due to the fact that this algorithm makes the iteration matrix symmetric [9], which indicates that the proposed SSOR-based precoding is more robust in practical systems. In addition, a low-complexity precoding scheme based on GS method was proposed in [8]. However, the proposed SSOR-based precoding can be considered as a generalization of the existing GS-based precoding, since when the relaxation parameter of SSOR-based precoding equals to 1, the SSOR-based precoding becomes GS-based precoding. Moreover, by introducing the relaxation parameter in SSOR-based precoding, the proposed scheme provides more flexibility for practical system design.

An important issue of SSOR-based precoding is the choice of the relaxation parameter ω, which influences the convergence rate of SSOR-based precoding as indicated by (9) and (10). In the next subsection, we will propose a simple approach to determine the optimal relaxation parameter for the proposed SSOR-based precoding.

C. Relaxation Parameter

According to [9], the optimal relaxation parameter ω_{opt} for SSOR method is

$$\omega_{opt} = \frac{2}{1+\sqrt{2(1-\rho[\mathbf{D}^{-1}(\mathbf{L}+\mathbf{L}^H)])}}, \tag{11}$$

where $\rho\,[\cdot]$ denotes the spectral radius of a matrix. However, the computation of $\rho\,[\cdot]$ is complicated. Moreover, if the matrix $\mathbf{P} = \mathbf{H}\mathbf{H}^H$ changes rapidly in fast time-varying channels, we need to recompute the optimal relaxation parameter ω_{opt}. Thus, directly using (11) to determine the optimal relaxation parameter is not preferred in practical massive MIMO systems. Compared with the classical SOR method, the convergence of SSOR method is less sensitive to the relaxation parameter due to the fact that SOR method is used twice, i.e., in the forward and reverse order, respectively [9]. This implies that we can use an approximation method to determine ω in a much simpler way for practical massive MIMO systems, which is provided in the following Lemma 1.

Lemma 1: In massive MIMO systems, the optimal relaxation parameter of SSOR-based precoding can be approximated by $\hat{\omega}_{opt} = \frac{2}{1+\sqrt{2(1-a)}}$, where $a = (1+\sqrt{\frac{K}{N}})^2 - 1$ only depends on the dimension (N, K) of massive MIMO systems.

Proof: From (8), we have

$$\begin{aligned}\rho[\mathbf{D}^{-1}(\mathbf{L}+\mathbf{L}^H)] &= \rho[\mathbf{D}^{-1}(\mathbf{P}-\mathbf{D})]\\ &= \rho[\mathbf{D}^{-1}\mathbf{P}-\mathbf{I}]\\ &= \rho[\mathbf{D}^{-1}\mathbf{P}]-1.\end{aligned} \tag{12}$$

For the typical flat Rayleigh fading channel matrix $\mathbf{H}$, the diagonal elements p_{mm} $(m = 1, 2, \cdots, K)$ of $\mathbf{P} = \mathbf{H}\mathbf{H}^H$ follow the $2N$-degree of freedom chi-square distribution [9]. By using the Chebyshev's inequality [9], we have

$$\begin{aligned}&\Pr\left(\left|\frac{p_{mm}-N}{N}\right| < \varepsilon\right)\\ &= \Pr[(1-\varepsilon)N < p_{mm} < (1+\varepsilon)N] \geq 1-\frac{1}{N\varepsilon^2},\end{aligned} \tag{13}$$

which indicates that when N grows large, the probability that $(1-\varepsilon)N < p_{mm} < (1+\varepsilon)N$ is valid will tend to 1. As a result, we can use N to reliably approximate p_{mm} in massive MIMO systems with a large number of BS antennas N. Accordingly, $\mathbf{D}^{-1}$ can be also approximated by $\frac{1}{N}\mathbf{I}$, then (12) becomes

$$\rho[\mathbf{D}^{-1}(\mathbf{L}+\mathbf{L}^H)] = \rho[\mathbf{D}^{-1}\mathbf{P}]-1 = \frac{1}{N}\rho[\mathbf{P}]-1. \tag{14}$$

Based on the random matrix theory, the spectral radius of $\mathbf{P}$ can be well approximated by [10]

$$\rho[\mathbf{P}] = N\left(1+\sqrt{\frac{K}{N}}\right)^2. \tag{15}$$

Substituting (15) into (14), we have

$$\hat{\omega}_{opt} = \frac{2}{1+\sqrt{2(1-a)}}, \tag{16}$$

where $a = (1+\sqrt{\frac{K}{N}})^2 - 1$. ■

Lemma 1 indicates that the deterministic relaxation parameter $\hat{\omega}_{opt}$ of the proposed SSOR-based precoding only depends on the number of BS antennas N and the number of users K, which are constant after the MIMO configuration has been fixed. Thus, we do not need to recompute $\hat{\omega}_{opt}$ when the channel $\mathbf{H}$ varies. Note that in realistic massive MIMO systems with limited N and K, there is a small gap between the exact optimal ω_{opt} and its approximation $\hat{\omega}_{opt}$. However, the approximated $\hat{\omega}_{opt}$ can still guarantee satisfying performance because SSOR-based precoding is not sensitive to the relaxation parameter, which will be verified by simulation results in Section IV.

D. Complexity Analysis

In this subsection, we provide the computational complexity analysis of the proposed SSOR-based precoding in terms of the required number of complex multiplications. For SSOR-based precoding, we can rewrite (9) as

$$t_k^{(i+1/2)} = t_k^{(i)} + \frac{\omega}{p_{ii}}\left(s_i - \sum_{j=1}^{k-1} p_{kj} t_j^{(k+1/2)} - \sum_{j=k}^{K} p_{kj} t_j^{(k)}\right), \tag{17}$$

where the subscript k denotes the kth element in a vector. For the fair comparison of complexity, the channel coherence interval T_c should be taken into account. As there are K elements in the vector $\mathbf{t}$ (where K is usually large in massive MIMO systems, e.g., $K = 32$ users can be simultaneously supported by a real demo of massive MIMO [11]), the complexity of ZF precoding within T_c is $\mathcal{O}(K^3) + T_c N K$, while the complexity of the proposed SSOR-based precoding is $T_c \mathcal{O}(K^2 + NK)$. As we have mentioned in Section III-B, the channel coherence interval T_c could be very small in fast time-varying channels [12]. For example, by considering the typical system parameters in current LTE-Advanced standard [13] and the user velocity of 100 km/h, the channel coherence time T_c is 7 OFDM symbols, which is smaller than K. Thus, the overall complexity of SSOR-based precoding is lower than that of ZF precoding.

TABLE I
COMPUTATIONAL COMPLEXITY COMPARISON

Iterative number	Neumann-based precoding [5]	TPE-based precoding [6]	Proposed SSOR-based precoding
$i = 2$	$3K^2 - K + T_c NK$	$(4 + T_c)NK$	$T_c(6K^2 + 3K + NK)$
$i = 3$	$K^3 + K^2 + T_c NK$	$(12 + T_c)NK$	$T_c(8K^2 + 3K + NK)$
$i = 4$	$2K^3 + T_c NK$	$(16 + T_c)NK$	$T_c(10K^2 + 3K + NK)$
$i = 5$	$3K^3 - K^2 + T_c NK$	$(20 + T_c)NK$	$T_c(12K^2 + 3K + NK)$

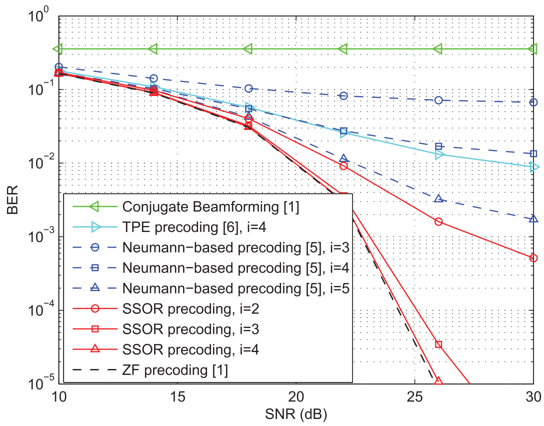

Fig. 1. BER performance comparison for the 128 × 16 massive MIMO system in Rayleigh fading channels.

In Table I, we compare the complexity of the proposed SSOR-based precoding with that of Neumann-based precoding [3] and TPE-based precoding [4] by taking the channel coherence interval T_c into account. Since the number of users K is large in massive MIMO systems, while T_c is small in fast time-varying channels, and the number of BS antennas N is usually much larger than the number of users K in massive MIMO systems, the proposed SSOR-based precoding enjoys a lower complexity than Neumann-based precoding and TPE-based precoding.

IV. Simulation Results

In this section, we evaluate the bit error rate (BER) performance of the proposed SSOR-based precoding compared with the recently proposed Neumann-based precoding. The BER performance of the classical ZF precoding with exact matrix inversion is also included as the benchmark for comparison. The configuration of massive MIMO systems is set to $N \times K = 128 \times 16$, and the modulation scheme is 64 QAM.

Fig. 1 shows the BER performance comparison in Rayleigh fading channels, where i denotes the number of iterations. From Fig. 1, we can first find that although conjugate beamforming [2] is considered to be near-optimal when the number of BS antennas goes to infinity in massive MIMO systems, the BER performance of conjugate beamforming suffers from severe performance loss due to the limited number of BS antennas in practical systems. Then, it is clear that the BER performance of SSOR-based precoding with $i = 2$ is even better than that of Neumann-based precoding [3] with $i = 4$ and TPE-based precoding [4] with $i = 4$. In addition, as the number of iterations increases, the performance of the proposed SSOR-based precoding improves fast. For example, the performance gap between ZF precoding and SSOR-based precoding is negligible when $i = 4$, which indicates that the proposed SSOR-based precoding can achieve the near-optimal performance within a small number of iterations.

V. Conclusions

In this letter, we propose a low-complexity SSOR-based precoding scheme to achieve the near-optimal performance of the classical ZF precoding with substantially reduced complexity. This is achieved by using SSOR method to iteratively approach the exact matrix inversion of large size in ZF precoding. Moreover, by exploiting the channel property of asymptotical orthogonality in massive MIMO systems, we propose a simple way to approximate the optimal relaxation parameter of the SSOR-based precoding, which only depends on the dimension of massive MIMO systems. Simulation results show that the proposed SSOR-based precoding outperforms the recently proposed precoding schemes, and approaches the near-optimal performance of ZF precoding in Rayleigh fading channels.

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