Direct Geolocation of Wideband Emitters Based on Delay and Doppler

Anthony J. Weiss, *Fellow, IEEE*

***Abstract*—The localization of a stationary transmitter using receivers mounted on fast moving platforms is considered. It is assumed that the transmitted radio signal is random with known statistics. The conventional approach is based on two steps. In the first step the time difference of arrival and the differential Doppler shift are measured and in the second step these measurements are used for geolocation. We advocate a direct position determination approach that proves to be more computationally efficient and more precise for weak signals than the conventional approach. The direct method is a single-step method that uses the same signals as the two-step approach but searches directly for the emitter position without first estimating intermediate parameters such as Doppler frequency and the time delay. A secondary but important result is a derivation of closed-form and compact expressions of the Cramér–Rao lower bound. All results are verified by Monte Carlo computer simulations.**

***Index Terms*—Differential Doppler, emitter location, maximum likelihood estimation.**

I. Introduction

PASSIVE geolocation of a stationary transmitter based on the delayed and Doppler shifted signal, observed by at least a single moving platform, is a well-known technique as can be concluded from [1]–[8]. Since the receivers location and velocity along their trajectory are known, the emitter location can be estimated.

Common methods use two steps for localization. The system first estimates the differential delay and differential Doppler frequency shift along the receivers trajectories. In the second step the system estimates the transmitter location based on the results obtained in the first step. The two step methods are not guaranteed to yield optimal location results since in the first step the differential delay and differential Doppler estimates are obtained by ignoring the constraint that all measurements must be consistent with a geolocation of a single point emitter. Thus, the lines of position (LOP) obtained from the delay/Doppler measurements are not guaranteed to intersect in a single geographical location. However, it can be shown that *asymptotically* (for large number of observations), the two-step methods are equivalent to the single step approach. The theory of the asymptotic equivalence, also known as the extended invariance principle (EXIP) is proved in [14] and [15]. We therefore conclude that the single step approach outperforms the two step methods for low signal-to-noise ratio (SNR) and/or short data records. This is demonstrated in this paper. Other advantages of the single step approach is the reduced computational complexity. In the conventional methods the Differential-Doppler and the differential-delay are estimated along the receivers track and finally the location of the transmitter is deduced. In the single-step method, only the transmitter location is estimated, directly from the observed signals. Therefore, our algorithm is not only more precise it is also easier to implement.

In a previous publication, we proposed a single-step solution for narrowband signals [11] based on Doppler shift only and assuming known or unknown nonrandom signals. Herein, we propose a maximum likelihood location estimation by using a single step approach for wideband random signals using both the Doppler effect and the relative delay.

II. Problem Formulation

Consider a stationary source transmitting a narrow band signal whose carrier frequency is f_c and its envelope is $s(t)$. The bandwidth of $s(t)$ is W and it satisfies $W \ll f_c$. Then, an appropriate signal model is $s(t)e^{\imath 2\pi f_c t}$. When the signal is observed by a moving sensor the observed signal becomes $s(\beta t)e^{\imath 2\pi f_c \beta t}$ which is an expansion or compression of the signal time scale, known as the Doppler effect. Let v be the sensor-source relative velocity and c the signal propagation velocity, then $\beta = 1 + v/c$. Since β is nearly one the effect on the slowly varying envelope will be negligible however the carrier frequency will be shifted considerably. Then the observed signal can be approximated by $s(t)e^{\imath 2\pi f_c \beta t}$. When the distance between the source and the sensor is considerable the signal will be delayed by τ and the observed signal becomes $s(t-\tau)e^{\imath 2\pi f_c \beta t}$ up to the complex constant $e^{-\imath 2\pi f_c \beta \tau}$ which has no effect on the signal amplitude, the signal delay or the signal frequency. This simple model will be used in the sequel.

Consider a stationary radio emitter and L receivers, synchronized in frequency and time, who are moving at a known speed that can be zero or more. The receivers intercept the transmitted signal at K short intervals along their track. Let $\mathbf{p}$ stand for the coordinates vector of the emitter and $\mathbf{p}_{\ell,k}$ and $\mathbf{v}_{\ell,k}$ denote the coordinates vector and the velocity vector of the ℓ receiver at the k interception interval. The complex signal, after frequency downconversion by a known carrier frequency f_c, observed by the ℓ receiver at the k interception interval at time t is given by

$$r_{\ell,k}(t) = s_k(t - \tau_{\ell,k})e^{\imath 2\pi f_{\ell,k} t} + w_{\ell,k}(t) \tag{1}$$

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The author is with the School of Electrical Engineering, Tel Aviv University, Tel Aviv 69978, Israel (e-mail: ajw@eng.tau.ac.il).

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where $-T/2 \leq t \leq T/2$ is the observation time interval of each interception, $s_k(t)$ is the observed signal envelope during the k interception interval, $\tau_{\ell,k}$ is the signal propagation time from the emitter to the receiver, $w_{\ell,k}(t)$ is a wide-sense stationary, white, zero mean, complex, Gaussian noise and $f_{\ell,k}$ is the observed Doppler frequency shift given by

$$f_{\ell,k} \triangleq f_c \mu_{\ell,k}(\mathbf{p}) \tag{2}$$

$$\mu_{\ell,k}(\mathbf{p}) \triangleq \frac{1}{c} \mathbf{v}_{\ell,k}^T (\mathbf{p} - \mathbf{p}_{\ell,k}) / \|\mathbf{p} - \mathbf{p}_{\ell,k}\| \tag{3}$$

where c is the signal propagation speed and $\|\cdot\|$ is the Euclidean norm. Similarly to [7] we assume that $\tau_{\ell,k} \ll T$. Further, note that we assumed that $\tau_{\ell,k}$ is quasi-constant during the observation time. That means that the observation time must be short enough to ensure that the platform position does not change more than the desired positioning error during the observation time.

Assume now that the signals $s_k(t)$ are realizations of a zero-mean, Gaussian process. We shall consider the waveform received by each sensor to be represented by its Fourier coefficients defined by

$$\tilde{r}_{\ell,k}(f_n) \triangleq \frac{1}{T} \int_{-T/2}^{T/2} r_{\ell,k}(t) e^{-\imath 2\pi f_n t}\, dt \tag{4}$$

where $f_n = n/T, n = 0, \pm 1, \pm 2, \ldots$ is the frequency associated with the n coefficient. The Fourier coefficients of $s_k(t)$ and of $w_{\ell,k}(t)$ are denoted by $\tilde{s}_k(f_n)$ and by $\tilde{w}_{\ell,k}(f_n)$, respectively. In the frequency domain (1) becomes, up to a multiplicative complex constant whose magnitude is 1,

$$\tilde{r}_{\ell,k}(f_n) = \tilde{s}_k(f_n - f_{\ell,k}) e^{-\imath 2\pi f_n \tau_{\ell,k}} + \tilde{w}_{\ell,k}(f_n). \tag{5}$$

Define the vectors

$$\begin{aligned}
\tilde{\mathbf{r}}_{\ell,k} &\triangleq [\tilde{r}_{\ell,k}(f_{-N}), \ldots, \tilde{r}_{\ell,k}(f_N)]^T \\
\tilde{\mathbf{w}}_{\ell,k} &\triangleq [\tilde{w}_{\ell,k}(f_{-N}), \ldots, \tilde{w}_{\ell,k}(f_N)]^T \\
\tilde{\mathbf{s}}_k &\triangleq [\tilde{s}_k(f_{-N}) \ldots, \tilde{s}_k(f_N)]^T \\
\tilde{\mathbf{A}}_{\ell,k} &\triangleq \operatorname{diag}\left\{e^{-\imath 2\pi \tau_{\ell,k} f_{-N}}, \ldots, e^{-\imath 2\pi \tau_{\ell,k} f_N}\right\}.
\end{aligned} \tag{6}$$

We get from (5) and (6) the relation

$$\tilde{\mathbf{r}}_{\ell,k} = \tilde{\mathbf{A}}_{\ell,k} \tilde{\mathbf{F}}_{\ell,k} \tilde{\mathbf{s}}_k + \tilde{\mathbf{w}}_{\ell,k} \tag{7}$$

where $\tilde{\mathbf{F}}_{\ell,k}$ is a cycle shift operator (matrix) that is used for down shifting. The product $\tilde{\mathbf{F}}_{\ell,k}\tilde{\mathbf{s}}_k$ shifts the vector $\tilde{\mathbf{s}}_k$ by $\lfloor T f_{\ell,k} \rfloor$ indices. For example, to obtain a matrix that down shifts by one index bring the last row of the identity matrix to be the first row. The first and second order moments of the noise vectors and the signal vectors are summarized by

$$\begin{aligned}
E\{\tilde{\mathbf{s}}_{\ell,k}\} &= E\{\tilde{\mathbf{w}}_{\ell,k}\} = 0, \quad \forall \ell, k \\
E\{\tilde{\mathbf{s}}_k \tilde{\mathbf{w}}_{\ell,j}^H\} &= 0, \quad \forall \ell, k, j \\
E\{\tilde{\mathbf{w}}_{\ell,k} \tilde{\mathbf{w}}_{i,j}^H\} &= \sigma^2 \mathbf{I} \delta_{\ell,i} \delta_{k,j} \\
E\{\tilde{\mathbf{s}}_k \tilde{\mathbf{s}}_j^H\} &= \mathbf{\Lambda} \delta_{k,j}
\end{aligned} \tag{8}$$

where $\mathbf{\Lambda}$ is a diagonal matrix whose main diagonal is the signal spectrum which is independent of the observations index $k \in \{1, \ldots K\}$. Obviously, $\mathbf{\Lambda}$ is diagonal only if the observation time, T, is much larger than the correlation time of the signal and as a result the Fourier coefficients are uncorrelated. In this paper we assume that the signal and noise are not correlated between interception points. Thus, (7) and (8) yield

$$\begin{aligned}
\tilde{\mathbf{R}}_{\ell,k,i,j} &\triangleq E\{\tilde{\mathbf{r}}_{\ell,k} \tilde{\mathbf{r}}_{i,j}^H\} \\
&= \tilde{\mathbf{A}}_{\ell,k} \tilde{\mathbf{F}}_{\ell,k} \mathbf{\Lambda} \tilde{\mathbf{F}}_{i,j}^H \tilde{\mathbf{A}}_{i,j}^H \delta_{k,j} + \sigma^2 \mathbf{I} \delta_{\ell,i} \delta_{k,j}.
\end{aligned} \tag{9}$$

Define the vectors and matrices

$$\begin{aligned}
\tilde{\mathbf{r}}_k &\triangleq \left[\tilde{\mathbf{r}}_{1,k}^T, \tilde{\mathbf{r}}_{2,k}^T, \ldots \tilde{\mathbf{r}}_{L,k}^T\right]^T \\
\tilde{\mathbf{r}} &\triangleq \left[\tilde{\mathbf{r}}_1^T, \tilde{\mathbf{r}}_2^T, \ldots \tilde{\mathbf{r}}_K^T\right]^T \\
\tilde{\mathbf{R}}_k &\triangleq E\left\{\tilde{\mathbf{r}}_k \tilde{\mathbf{r}}_k^H\right\} \\
\tilde{\mathbf{R}} &\triangleq E\{\tilde{\mathbf{r}} \tilde{\mathbf{r}}^H\}.
\end{aligned} \tag{10}$$

The Gaussian probability density function of the data is given by

$$f_{\tilde{\mathbf{r}}}(\tilde{\mathbf{r}} \,|\, \mathbf{p}) = (\pi |\tilde{\mathbf{R}}|)^{-1} \exp\{-\tilde{\mathbf{r}}^H \tilde{\mathbf{R}}^{-1} \tilde{\mathbf{r}}\} \tag{11}$$

and the negative log-likelihood is given, up to an additive constant, by

$$\begin{aligned}
L_f(\mathbf{p}) &= \log\{|\tilde{\mathbf{R}}|\} + \tilde{\mathbf{r}}^H \tilde{\mathbf{R}}^{-1} \tilde{\mathbf{r}} \\
&= \log\left\{\prod_{k=1}^{K} |\tilde{\mathbf{R}}_k|\right\} + \sum_{k=1}^{K} \tilde{\mathbf{r}}_k^H \tilde{\mathbf{R}}_k^{-1} \tilde{\mathbf{r}}_k.
\end{aligned} \tag{12}$$

Using the definition

$$\tilde{\mathbf{B}}_k \triangleq \left[\tilde{\mathbf{F}}_{1,k}^H \tilde{\mathbf{A}}_{1,k}^H, \ldots \tilde{\mathbf{F}}_{L,k}^H \tilde{\mathbf{A}}_{L,k}^H\right]^H, \tag{13}$$

we get from (9) and (10) the relation

$$\tilde{\mathbf{R}}_k = \tilde{\mathbf{B}}_k \mathbf{\Lambda} \tilde{\mathbf{B}}_k^H + \sigma^2 \mathbf{I}. \tag{14}$$

We use the Woodbury matrix identity [12] and the matrix determinant lemma [13] to obtain

$$\tilde{\mathbf{R}}_k^{-1} = \sigma^{-2} \left(\mathbf{I} - \tilde{\mathbf{B}}_k \left(\sigma^2 \mathbf{\Lambda}^{-1} + \tilde{\mathbf{B}}_k^H \tilde{\mathbf{B}}_k\right)^{-1} \tilde{\mathbf{B}}_k^H\right) \tag{15}$$

$$|\tilde{\mathbf{R}}_k| = \left|\mathbf{\Lambda}^{-1} + \sigma^{-2} \tilde{\mathbf{B}}_k^H \tilde{\mathbf{B}}_k\right| |\mathbf{\Lambda}| |\sigma^2 \mathbf{I}|. \tag{16}$$

Further simplification is obtained by noting that

$$\tilde{\mathbf{B}}_k^H \tilde{\mathbf{B}}_k = \sum_{\ell=1}^{L} \tilde{\mathbf{F}}_{\ell,k}^H \tilde{\mathbf{A}}_{\ell,k}^H \tilde{\mathbf{A}}_{\ell,k} \tilde{\mathbf{F}}_{\ell,k} = L\mathbf{I}. \tag{17}$$

This result indicates that $|\mathbf{R}_k|$ is independent of the emitter location. Define

$$\begin{aligned}
\mathbf{\Gamma} &\triangleq \left(\sigma^2 \mathbf{\Lambda}^{-1} + \tilde{\mathbf{B}}_k^H \tilde{\mathbf{B}}_k\right)^{-1} \\
&= \operatorname{diag}\left\{\frac{\lambda_1}{(\sigma^2 + L\lambda_1)}, \ldots \frac{\lambda_{N_1}}{(\sigma^2 + L\lambda_{N_1})}\right\}
\end{aligned} \tag{18}$$

where $\lambda_i = \mathbf{\Lambda}_{i,i}$. Thus, (15) becomes

$$\tilde{\mathbf{R}}_k^{-1} = \sigma^{-2}\left(\mathbf{I} - \tilde{\mathbf{B}}_k \mathbf{\Gamma} \tilde{\mathbf{B}}_k^H\right). \quad (19)$$

The cost function in (12) can be simplified and rewritten as

$$C_r(\mathbf{p}) = \sum_{k=1}^{K} \tilde{\mathbf{r}}_k^H \tilde{\mathbf{R}}_k^{-1} \tilde{\mathbf{r}}_k = \sigma^{-2} \sum_{k=1}^{K} \tilde{\mathbf{r}}_k^H \left(\mathbf{I} - \tilde{\mathbf{B}}_k \mathbf{\Gamma} \tilde{\mathbf{B}}_k^H\right) \tilde{\mathbf{r}}_k. \quad (20)$$

Instead of finding the minimum of (20) we can find the maximum of

$$\begin{aligned}\tilde{C}_r(\mathbf{p}) &= \sum_{k=1}^{K} \tilde{\mathbf{r}}_k^H \tilde{\mathbf{B}}_k \mathbf{\Gamma} \tilde{\mathbf{B}}_k^H \tilde{\mathbf{r}}_k \\ &= \sum_{k=1}^{K} \left\| \mathbf{\Gamma}^{1/2} \sum_{\ell=1}^{L} \tilde{\mathbf{F}}_{\ell,k}^H(\mathbf{p}) \tilde{\mathbf{A}}_{\ell,k}^H(\mathbf{p}) \tilde{\mathbf{r}}_{\ell,k} \right\|^2. \end{aligned} \quad (21)$$

where the dependence of both $\tilde{\mathbf{F}}_{\ell,k}^H$ and $\tilde{\mathbf{A}}_{\ell,k}^H$ on $\mathbf{p}$ is shown explicitly. The estimated emitter location is then given by

$$\hat{\mathbf{p}} = \arg\max_{\mathbf{p}} \{\tilde{C}_r(\mathbf{p})\}. \quad (22)$$

A possible algorithm is displayed in Algorithm 1 (A Possible Implementation of the DPD Algorithm).

```
Define the area of interest and determine a suitable grid
of locations p_1, p_2 ··· p_g.
for j = 1 to g do
    Set C̃_r(p_j) = 0
    for k = 1 to K do
        Set G = 0
        for ℓ = 1 to L do
            Evaluate the delay, τ_{ℓ,k}, and Doppler, f_{ℓ,k},
            for a transmitter at p_j.
            Evaluate Ã_{ℓ,k}, F̃_{ℓ,k}
            Evaluate G = G + F̃^H_{ℓ,k} Ã^H_{ℓ,k} r̃_{ℓ,k}
        end
        Let C̃(p_j) = C̃(p_j) + ||Γ^{1/2} G||^2
    end
end
Find the grid point for which C̃ is the biggest. This grid
point is the estimated position.
```

In order to obtain some insight, consider the case of only $L = 2$ receivers and a signal with flat spectrum so that $\mathbf{\Gamma}$ is a scaled identity matrix. In this case, ignoring constant terms, the cost function in (21) can be replaced by,

$$\tilde{C}_r(\mathbf{p}) = \sum_{k=1}^{K} \Re\left\{\tilde{\mathbf{r}}_{1,k}^H \tilde{\mathbf{A}}_{1,k} \tilde{\mathbf{F}}_{1,k} \tilde{\mathbf{F}}_{2,k}^H \tilde{\mathbf{A}}_{2,k}^H \tilde{\mathbf{r}}_{2,k}\right\} \quad (23)$$

where $\Re\{\cdot\}$ stands for the real part. Note that the term in curly brackets can be approximated by

$$\begin{aligned}&\tilde{\mathbf{r}}_{1,k}^H \tilde{\mathbf{A}}_{1,k} \tilde{\mathbf{F}}_{1,k} \tilde{\mathbf{F}}_{2,k}^H \tilde{\mathbf{A}}_{2,k}^H \tilde{\mathbf{r}}_{2,k} \\ &\quad \cong \int \tilde{r}_{1,k}^*(f + f_{1,k}) \tilde{r}_{2,k}(f + f_{2,k}) e^{i2\pi[\tau_{2,k} - \tau_{1,k}]f}\, df \end{aligned} \quad (24)$$

where the dependence of $\tau_{i,k}$ and $f_{i,k}$ on $\mathbf{p}$ is suppressed. The integral above is recognized as the well known complex ambiguity function. According to (22)–(24), the estimated position, $\hat{\mathbf{p}}$, maximizes the sum, over the interception instances, of the received signals cross-correlations. Note in passing that the proposed algorithm requires a two-dimensional search for the emitter location if its altitude is known or three dimensional search in the general case. However, the two-step methods require first an estimate of differential-delay and differential-Doppler for all intercept intervals and all the receivers and only then a search for the emitter location. Obviously, (21) and in (23) can also be used for estimating the differential Doppler and the differential delay.

III. Cramér–Rao Bound for Random Signals

The Cramér–Rao Bound is a lower theoretical bound on the covariance of any unbiased estimator. The bound is given by the inverse of the Fisher information matrix (FIM). According to [16], for zero-mean, complex, Gaussian data vectors with covariance matrix $\mathbf{R}$ the elements of the FIM are given by

$$[\mathbf{J}]_{i,j} = \mathrm{tr}\left\{\mathbf{R}^{-1} \frac{\partial \mathbf{R}}{\partial \psi_i} \mathbf{R}^{-1} \frac{\partial \mathbf{R}}{\partial \psi_j}\right\} \quad (25)$$

where ψ_i is the i element of the unknown parameter vector. In our case, the parameter vector is the transmitter coordinates vector. Thus, $\psi_1 = x, \psi_2 = y$. Assume that the signal spectrum is flat and its bandwidth is W [rad/s]. We show in the Appendix that the 2 FIM is given by

$$\begin{aligned}\mathbf{J}_{1,1} &= \Phi\, \mathrm{tr}\left\{W^2 \left[\mathbf{V}_\tau^{(x)}\right]^T \mathbf{V}_\tau^{(x)} + (2\pi T)^2 \left[\mathbf{V}_f^{(x)}\right]^T \mathbf{V}_f^{(x)}\right\} \\ \mathbf{J}_{2,2} &= \Phi\, \mathrm{tr}\left\{W^2 \left[\mathbf{V}_\tau^{(y)}\right]^T \mathbf{V}_\tau^{(y)} + (2\pi T)^2 \left[\mathbf{V}_f^{(y)}\right]^T \mathbf{V}_f^{(y)}\right\} \\ \mathbf{J}_{1,2} &= \Phi\, \mathrm{tr}\left\{W^2 \left[\mathbf{V}_\tau^{(x)}\right]^T \mathbf{V}_\tau^{(y)} + (2\pi T)^2 \left[\mathbf{V}_f^{(x)}\right]^T \mathbf{V}_f^{(y)}\right\}. \end{aligned} \quad (26)$$

where $\mathbf{V}_\tau^{(x)}$ and $\mathbf{V}_\tau^{(y)}$ represent the derivative of the delays w.r.t. x and the derivative of the delays w.r.t. y, respectively. Similarly, $\mathbf{V}_f^{(x)}$ and $\mathbf{V}_f^{(y)}$ represent the derivative of the Doppler shift w.r.t. x and the derivative of the Doppler shifts w.r.t. y, respectively. See (73) and (81) for more details. Further,

$$\Phi \triangleq \frac{WT\,\mathrm{SNR}}{12\pi}. \quad (27)$$

This result is pleasing since $(12\pi)/(W^3 T\,\mathrm{SNR})$ is recognized as the Cramér–Rao lower bound on time delay estimation [20], [21] in the absence of Doppler shift and $3/(\pi W T^3 \mathrm{SNR})$ is recognized as the Cramér–Rao lower bound on differential Doppler estimation [4] in the absence of delay.

IV. Time and Frequency Synchronization Errors

So far, we have assumed perfect time and frequency synchronization of all receivers. That means that the clocks at all platforms have exactly the same time and the same frequency. In practice this is impossible even if GPS and atomic clocks are used. It is therefore of interest to examine the sensitivity of the localization precision to time/frequency errors. Although a brute force small error analysis is possible we choose here to obtain the result using asymptotic reasoning.

As explained in the introduction, the two-step approach and the single-step approach are *asymptotically* equivalent. Thus, the error covariance is asymptotically the same. The two-step approach in our case consist of first estimating the differential Doppler (DD) and the time difference (TD) and then using the results for localization. The estimation error of DD and TD is affected directly by the synchronization errors of the receivers. Therefore, the TD estimation error variance is obtained by adding to the error variance caused by the noise the variance of synchronization errors

$$\sigma^2_{\mathrm{TD}} \triangleq \frac{12\pi}{W^3 T\,\mathrm{SNR}} + \sigma^2_\tau, \tag{28}$$

and the DD estimation error is similarly given by

$$\sigma^2_{\mathrm{DD}} \triangleq \frac{3}{\pi W T^3 \mathrm{SNR}} + \sigma^2_f \tag{29}$$

where σ^2_τ and σ^2_f are the variance of the time synchronization error between receivers and σ^2_f is the variance of the frequency synchronization errors.

Finally, the localization error covariance is given by the inverse of the matrix whose elements are defined by

$$\begin{aligned}
\tilde{\mathbf{J}}_{1,1} &= \mathrm{tr}\left\{\sigma^{-2}_{\mathrm{TD}}\left[\mathbf{V}^{(x)}_\tau\right]^T \mathbf{V}^{(x)}_\tau + \sigma^{-2}_{\mathrm{DD}}\left[\mathbf{V}^{(x)}_f\right]^T \mathbf{V}^{(x)}_f\right\} \\
\tilde{\mathbf{J}}_{2,2} &= \mathrm{tr}\left\{\sigma^{-2}_{\mathrm{TD}}\left[\mathbf{V}^{(y)}_\tau\right]^T \mathbf{V}^{(y)}_\tau + \sigma^{-2}_{\mathrm{DD}}\left[\mathbf{V}^{(y)}_f\right]^T \mathbf{V}^{(y)}_f\right\} \\
\tilde{\mathbf{J}}_{1,2} &= \mathrm{tr}\left\{\sigma^{-2}_{\mathrm{TD}}\left[\mathbf{V}^{(x)}_\tau\right]^T \mathbf{V}^{(y)}_\tau + \sigma^{-2}_{\mathrm{DD}}\left[\mathbf{V}^{(x)}_f\right]^T \mathbf{V}^{(y)}_f\right\}.
\end{aligned} \tag{30}$$

This concludes this section.

V. Numerical Examples

In this section, we provide a numerical example of the algorithm performance. We simulated two platforms whose speed is 300 m/s and a stationary transmitter as depicted in Fig. 1. The distance between the platforms is 1000 m along the x axis and 50 m along the y axis. The transmitter is located at coordinates 2000, 2000 m. The first receiver intercepts the signal when it is located at (0, 0),(1000, 0) and (2000, 0) m. The second receiver intercepts the signal when it is located at (1000, 50), (2000, 50), and (3000, 50) m. The transmitter carrier frequency is 1 GHz, and the signal bandwidth is 262.144 KHz with flat spectrum. The signal observation time at each interception point is 3.9 ms.

Fig. 2 shows the root-mean-square miss distance as a function of SNR. The miss distance is the distance between the true location and the estimated location. The data is based on 200 Monte Carlo trials for each data point. The performance of the advocated method (single-step a.k.a. DPD), the performance of the conventional, two-step method, and the Cramér-Rao lower bound are all shown in Fig. 2. Observe that for high SNR, the DPD root-mean-square miss distance is the same as that of the two-step method. However, for low SNR, the DPD provides much better performance.

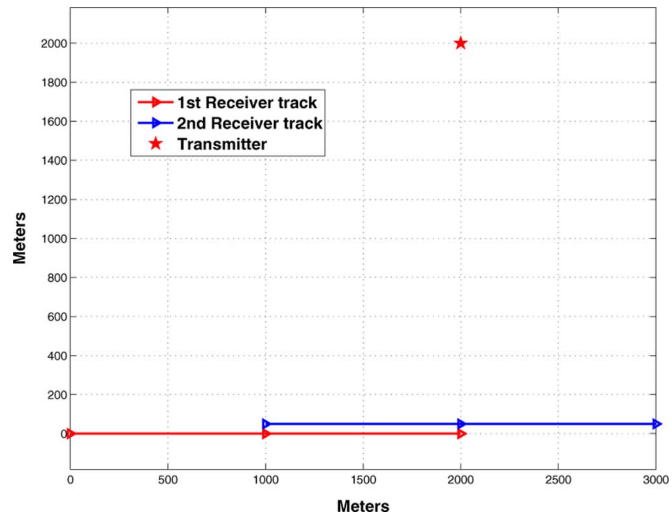

Fig. 1. Receivers track and emitter location.

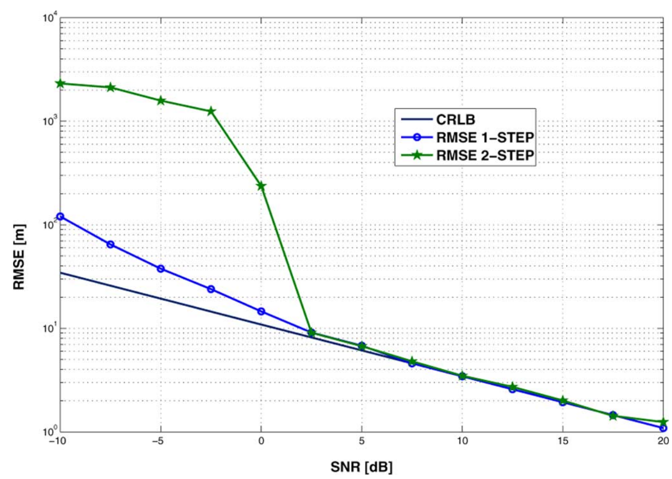

Fig. 2. RMSE of the DPD method (single step), the two-step method, and the Cramér–Rao lower bound versus SNR.

Fig. 3 shows the root-mean-square miss distance as a function of the receivers' frequency error. In order to demonstrate the effect of the frequency error, we had to suppress the effect of the additive noise and the timing error on the localization error. It should be realized that, at moderate SNR, the error is dominated by the noise, and therefore we used the relatively high SNR of 30 dB. In order to eliminate the effect of the timing error, the receivers' spacing was reduced to 1 m so the location was based only on the Doppler effect and not on the differential time of arrival. Both the DPD and the two-step conventional method had similar performance. The simulation agrees with the theory for small errors and diverges from the prediction when the errors are large.

Fig. 4 shows the root-mean-square miss distance as a function of the receivers' timing error. In order to demonstrate the timing error effect on the localization error, we suppressed the noise effect by choosing the high SNR of 30 dB. In order to eliminate

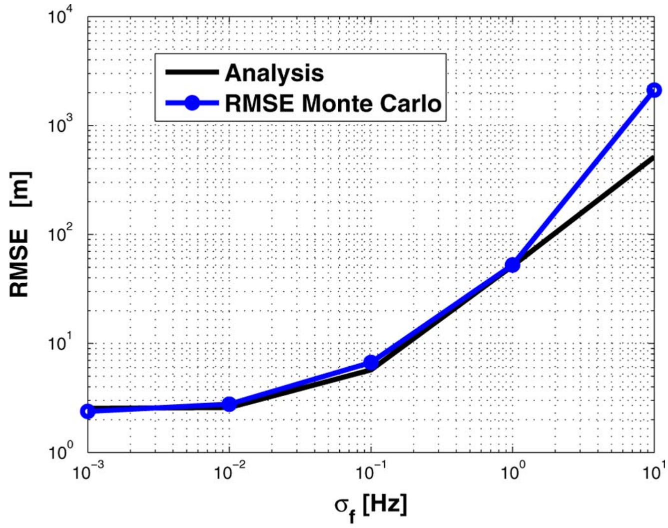

Fig. 3. RMSE of the DPD method and the theoretically predicted accuracy versus the receivers frequency error standard deviation.

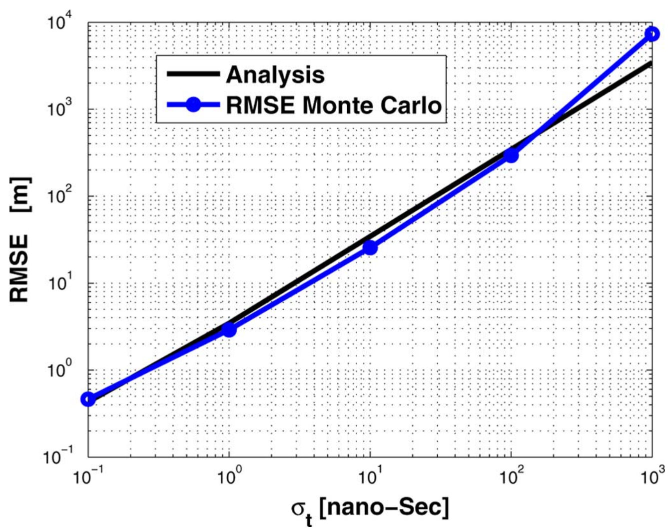

Fig. 4. RMSE of the DPD method (single step), the two-step method and the theoretically predicted accuracy versus the receivers timing error standard deviation.

the frequency error effect, we reduced the speed of the receivers to 10 m/s so the localization was based only on the time difference of arrival and not on the differential Doppler. Both the DPD and the two-step conventional method had similar performance. The simulation agrees with the theory for small errors and diverges from the prediction when the errors are large.

VI. Conclusion

In this paper, we proposed an algorithm for geolocation of an emitter observed by moving receivers. The transmitted signal is modeled as a random Gaussian signal and the localization is based on the delay and Doppler frequency shift that affect the observed signal. The proposed algorithm is the exact maximum likelihood. It is a single-step estimator as opposed to two-step methods that estimate the delay and the frequency shift in a first step and estimate the location in a second step. We demonstrated that the proposed method is superior to two step methods for low signal to noise ratio. We also presented closed-form expressions for the Cramér–Rao lower bound. The expressions are simple and intuitive. We also provided sensitivity analysis for timing and frequency errors between receivers. An extension to receivers location/velocity errors is straightforward.

Appendix A
Derivation of the Cramér–Rao Bound

The derivation of the CRLB is simpler in the time domain. Thus, assume that we collect N time samples of (1) and define

$$\begin{aligned}\mathbf{r}_{\ell,k} &\triangleq [r_{\ell,k}(t_1),\ldots,r_{\ell,k}(t_N)]^T\\ \mathbf{w}_{\ell,k} &\triangleq [w_{\ell,k}(t_1),\ldots,w_{\ell,k}(t_N)]^T\\ \mathbf{s}_k &\triangleq [s_k(t_1)\ldots,s_k(t_N)]^T,\\ \mathbf{A}_{\ell,k} &\triangleq \mathrm{diag}\left\{e^{\imath 2\pi f_{\ell,k}t_1},\ldots,e^{\imath 2\pi f_{\ell,k}t_N}\right\}.\end{aligned} \tag{31}$$

From (1), we get

$$\mathbf{r}_{\ell,k}=\mathbf{A}_{\ell,k}\mathbf{F}_{\ell,k}\mathbf{s}_k+\mathbf{w}_{\ell,k} \tag{32}$$

where $\mathbf{F}_{\ell,k}$ is a down shift operator. The product $\mathbf{F}_{\ell,k}\mathbf{s}_k$ shifts the vector $\mathbf{s}_k$ by $\lfloor \tau_{\ell,k}/T_s \rfloor$ indexes where T_s is the sampling period. From (32) we get the cross-correlation matrix

$$\begin{aligned}\mathbf{R}_{\ell,k,i,j} &\triangleq E\left\{\mathbf{r}_{\ell,k}\mathbf{r}_{i,j}^H\right\}\\ &= \mathbf{A}_{\ell,k}\mathbf{F}_{\ell,k}\mathbf{P}\mathbf{F}_{i,j}^H\mathbf{A}_{i,j}^H\delta_{k,j}+\sigma^2\mathbf{I}\delta_{\ell,i}\delta_{k,j}\end{aligned} \tag{33}$$

where we assumed that the noise is white and $\mathbf{P}\triangleq E\{\mathbf{s}_k\mathbf{s}_k^H\}$ is the signal covariance matrix. Define the vectors and matrices

$$\begin{aligned}\mathbf{r}_k &\triangleq \left[\mathbf{r}_{1,k}^T,\mathbf{r}_{2,k}^T,\ldots\mathbf{r}_{L,k}^T\right]^T\\ \mathbf{r} &\triangleq \left[\mathbf{r}_1^T,\mathbf{r}_2^T,\ldots\mathbf{r}_K^T\right]^T\\ \mathbf{R}_k &\triangleq E\left\{\mathbf{r}_k\mathbf{r}_k^H\right\}\\ \mathbf{R} &\triangleq E\{\mathbf{r}\mathbf{r}^H\},\\ \mathbf{B}_k &\triangleq \left[\mathbf{F}_{1,k}^H\mathbf{A}_{1,k}^H,\ldots\mathbf{F}_{L,k}^H\mathbf{A}_{L,k}^H\right]^H.\end{aligned} \tag{34}$$

From (34), we now get

$$\mathbf{R}_k=\mathbf{B}_k\mathbf{P}\mathbf{B}_k^H+\sigma^2\mathbf{I}. \tag{35}$$

The matrix $\mathbf{R}$ is block diagonal, with K blocks, and therefore (25) becomes

$$[\mathbf{J}]_{i,j}=\sum_{k=1}^{K}\mathrm{tr}\left\{\mathbf{R}_k^{-1}\frac{\partial\mathbf{R}_k}{\partial\psi_i}\mathbf{R}_k^{-1}\frac{\partial\mathbf{R}_k}{\partial\psi_j}\right\}. \tag{36}$$

Using the chain rule, we can write

$$\frac{\partial\mathbf{R}_k}{\partial x}=\sum_{\ell=1}^{L}\frac{\partial\mathbf{R}_k}{\partial f_{\ell,k}}\cdot\frac{\partial f_{\ell,k}}{\partial x}+\frac{\partial\mathbf{R}_k}{\partial\tau_{\ell,k}}\cdot\frac{\partial\tau_{\ell,k}}{\partial x}. \tag{37}$$

Note that

$$\frac{\partial \mathbf{R}_k}{\partial f_{\ell,k}} = \frac{\partial \mathbf{B}_k}{\partial f_{\ell,k}} \mathbf{P}\mathbf{B}_k^H + \mathbf{B}_k \mathbf{P} \frac{\partial \mathbf{B}_k^H}{\partial f_{\ell,k}}, \tag{38}$$

and

$$\begin{aligned}\frac{\partial \mathbf{B}_k}{\partial f_{\ell,k}} &= i2\pi \left[\ldots \mathbf{0}, \ldots \mathbf{F}_{\ell,k}^T \mathbf{A}_{\ell,k}^T \mathbf{T}, \ldots \mathbf{0}, \ldots\right]^T \\ &= i2\pi(\mathbf{e}_\ell \otimes \mathbf{T}\mathbf{A}_{\ell,k}\mathbf{F}_{\ell,k})\end{aligned} \tag{39}$$

where $\mathbf{T} \triangleq \mathrm{diag}\{t_1, t_2, \ldots t_N\}$ and $\mathbf{e}_\ell$ is a $L \times 1$ vector of zeros except for the ℓ entry which is one. Therefore, (38) and (39) yield

$$\frac{\partial \mathbf{R}_k}{\partial f_{\ell,k}} = i2\pi \left(\mathbf{e}_\ell \otimes \mathbf{T}\mathbf{A}_{\ell,k}\mathbf{F}_{\ell,k}\mathbf{P}\mathbf{B}_k^H\right) - i2\pi \left(\mathbf{e}_\ell^T \otimes \mathbf{B}_k \mathbf{P} \mathbf{F}_{\ell,k}^H \mathbf{A}_{\ell,k}^H \mathbf{T}\right). \tag{40}$$

Define now the block diagonal matrix

$$\mathbf{A}_k \triangleq \mathrm{diag}\{\mathbf{A}_{1,k}, \ldots \mathbf{A}_{L,k}\}, \tag{41}$$

and the $NL \times NL$ block matrix, $\bar{\mathbf{P}}_k$, whose i,j block is given by

$$\begin{aligned}[\bar{\mathbf{P}}_k]_{i,j} \triangleq E\{&[s_k(t_1 - \tau_{i,k}), \ldots s_k(t_N - \tau_{i,k})]^T \\ &\times [s_k(t_1 - \tau_{j,k}), \ldots s_k(t_N - \tau_{j,k})]^*\}\end{aligned} \tag{42}$$

and the m,n element of this block is

$$\left[[\bar{\mathbf{P}}_k]_{i,j}\right]_{m,n} = R(t_m - t_n - \tau_{i,k} + \tau_{j,k}). \tag{43}$$

Note that $\mathbf{R}_k = \mathbf{A}_k \bar{\mathbf{P}}_k \mathbf{A}_k^H + \sigma^2 \mathbf{I}$ and

$$\frac{\partial \mathbf{R}_k}{\partial \tau_{\ell,k}} = \mathbf{A}_k \frac{\partial \bar{\mathbf{P}}_k}{\partial \tau_{\ell,k}} \mathbf{A}_k^H \tag{44}$$

and

$$\begin{aligned}\frac{\partial \left[[\bar{\mathbf{P}}_k]_{i,j}\right]_{m,n}}{\partial \tau_{\ell,k}} &= -\dot{R}(t_m - t_n - \tau_{i,k} + \tau_{j,k})\delta_{\ell,i} \\ &\quad + \dot{R}(t_m - t_n - \tau_{i,k} + \tau_{j,k})\delta_{\ell,j} \\ &= -\delta_{\ell,i}\left[\mathbf{F}_{i,k}\dot{\mathbf{P}}\mathbf{F}_{j,k}^H\right]_{m,n} \\ &\quad + \delta_{\ell,j}\left[\mathbf{F}_{i,k}\dot{\mathbf{P}}\mathbf{F}_{j,k}^H\right]_{m,n}.\end{aligned} \tag{45}$$

where $\dot{R}(\tau)$ is the derivative of $R(\tau)$ with respect to τ, and we used

$$\dot{\mathbf{P}}_{m,n} \triangleq \dot{R}(t_m - t_n). \tag{46}$$

Finally, (44) becomes

$$\frac{\partial \mathbf{R}_k}{\partial \tau_{\ell,k}} = -\mathbf{e}_\ell \otimes \mathbf{A}_{\ell,k}\mathbf{F}_{\ell,k}\dot{\mathbf{P}}\mathbf{B}_k^H + \mathbf{e}_\ell^T \otimes \mathbf{B}_k \dot{\mathbf{P}} \mathbf{F}_{\ell,k}^H \mathbf{A}_{\ell,k}^H \tag{47}$$

where $\dot{\mathbf{P}}$ is a matrix whose elements are given in (46). Define now the block diagonal version of $\mathbf{B}_k$ given by

$$\mathbf{G}_k \triangleq \mathrm{diag}\{\mathbf{A}_{1,k}\mathbf{F}_{1,k}, \ldots \mathbf{A}_{L,k}\mathbf{F}_{L,k}\}. \tag{48}$$

Note that

$$\mathbf{G}_k^H \mathbf{B}_k = \mathbf{1}_L \otimes \mathbf{I}_N \triangleq \mathbf{U}. \tag{49}$$

We can now replace $\mathbf{R}_k$ with $\mathbf{G}_k^H \mathbf{R}_k \mathbf{G}_k$ without changing the FIM. We get

$$\mathbf{G}_k^H \mathbf{R}_k \mathbf{G}_k = \mathbf{U}\mathbf{P}\mathbf{U}^T + \sigma^2 \mathbf{I}_{NL} \tag{50}$$

and

$$\begin{aligned}\left(\mathbf{G}_k^H \mathbf{R}_k \mathbf{G}_k\right)^{-1} &= \sigma^{-2}\left(\mathbf{I}_{NL} - \mathbf{U}(\sigma^2 \mathbf{P}^{-1} + L\mathbf{I}_N)^{-1}\mathbf{U}^T\right) \\ &= \sigma^{-2}\left(\mathbf{I}_{NL} - \mathbf{1}_L \mathbf{1}_L^T \otimes \mathbf{Z}\right)\end{aligned} \tag{51}$$

where

$$\mathbf{Z} \triangleq \left(\sigma^2 \mathbf{P}^{-1} + L\mathbf{I}_N\right)^{-1}. \tag{52}$$

Also from (40)

$$\begin{aligned}\mathbf{G}_k^H \frac{\partial \mathbf{R}_k}{\partial f_{\ell,k}} \mathbf{G}_k &= i2\pi\left(\mathbf{e}_\ell \mathbf{1}_L^T \otimes \tilde{\mathbf{T}}_{\ell,k}\mathbf{P} - \mathbf{1}_L \mathbf{e}_\ell^T \otimes \mathbf{P}\tilde{\mathbf{T}}_{\ell,k}\right) \\ &\simeq i2\pi\left(\mathbf{e}_\ell \mathbf{1}_L^T - \mathbf{1}_L \mathbf{e}_\ell^T\right) \otimes \tilde{\mathbf{T}}_{\ell,k}\mathbf{P}\end{aligned} \tag{53}$$

where $\tilde{\mathbf{T}}_{\ell,k} \triangleq \mathbf{F}_{\ell,k}^H \mathbf{T} \mathbf{F}_{\ell,k}$ is a diagonal matrix, and the last approximation holds for large time-bandwidth product [4]. Further, from (47), we have

$$\begin{aligned}\mathbf{G}_k^H \frac{\partial \mathbf{R}_k}{\partial \tau_{\ell,k}} \mathbf{G}_k &= -\mathbf{e}_\ell \mathbf{1}_L^T \otimes \dot{\mathbf{P}} + \mathbf{1}_L \mathbf{e}_\ell^T \otimes \dot{\mathbf{P}} \\ &= \left(\mathbf{1}_L \mathbf{e}_\ell^T - \mathbf{e}_\ell \mathbf{1}_L^T\right) \otimes \dot{\mathbf{P}}.\end{aligned} \tag{54}$$

Thus, (51) and (54) yield

$$\begin{aligned}&\left(\mathbf{G}_k^H \mathbf{R}_k \mathbf{G}_k\right)^{-1} \mathbf{G}_k^H \frac{\partial \mathbf{R}_k}{\partial \tau_{\ell,k}} \mathbf{G}_k \\ &\quad = \sigma^{-2}\left\{\left(\mathbf{1}_L \mathbf{e}_\ell^T - \mathbf{e}_\ell \mathbf{1}_L^T\right) \otimes \dot{\mathbf{P}} + \left(\mathbf{1}_L \mathbf{1}_L^T - L\mathbf{1}_L \mathbf{e}_\ell^T\right) \otimes \mathbf{Z}\dot{\mathbf{P}}\right\}.\end{aligned} \tag{55}$$

Similarly, (51) and (53) yield

$$\begin{aligned}&\left(\mathbf{G}_k^H \mathbf{R}_k \mathbf{G}_k\right)^{-1} \mathbf{G}_k^H \frac{\partial \mathbf{R}_k}{\partial f_{\ell,k}} \mathbf{G}_k \\ &\quad = -i2\pi\sigma^{-2}\left\{\left(\mathbf{1}_L \mathbf{e}_\ell^T - \mathbf{e}_\ell \mathbf{1}_L^T\right) \otimes \tilde{\mathbf{T}}_{\ell,k}\mathbf{P} \right. \\ &\qquad \left. + \left(\mathbf{1}_L \mathbf{1}_L^T - L\mathbf{1}_L \mathbf{e}_\ell^T\right) \otimes \mathbf{Z}\tilde{\mathbf{T}}_{\ell,k}\mathbf{P}\right\}.\end{aligned} \tag{56}$$

Define

$$\mathbf{W}_{\ell,k}^{(x)} \triangleq \dot{\mathbf{P}}\dot{\tau}_{\ell,k}^{(x)} - i2\pi \tilde{\mathbf{T}}_{\ell,k}\mathbf{P}\dot{f}_{\ell,k}^{(x)} \tag{57}$$

where $\dot{\tau}_{\ell,k}^{(x)}, \dot{f}_{\ell,k}^{(x)}$ are the derivatives of $\tau_{\ell,k}$ and $f_{\ell,k}$ with respect to x. Combining (37), (55), (56), and (57) yields

$$\begin{aligned}&\left(\mathbf{G}_k^H \mathbf{R}_k \mathbf{G}_k\right)^{-1} \mathbf{G}_k^H \frac{\partial \mathbf{R}_k}{\partial x} \mathbf{G}_k \\ &\quad = \sigma^{-2}\sum_{\ell=1}^{L}\left(\mathbf{1}_L \mathbf{e}_\ell^T - \mathbf{e}_\ell \mathbf{1}_L^T\right) \otimes \mathbf{W}_{\ell,k}^{(x)} \\ &\qquad + \left(\mathbf{1}_L \mathbf{1}_L^T - L\mathbf{1}_L \mathbf{e}_\ell^T\right) \otimes \mathbf{Z}\mathbf{W}_{\ell,k}^{(x)}.\end{aligned} \tag{58}$$

Define

$$\mathbf{U}_\ell \triangleq \mathbf{1}_L \mathbf{e}_\ell^T - \mathbf{e}_\ell \mathbf{1}_L^T \tag{59}$$
$$\mathbf{O}_\ell \triangleq \mathbf{1}_L \mathbf{1}_L^T - L \mathbf{1}_L \mathbf{e}_\ell^T. \tag{60}$$

Squaring (58) and using the definitions (59) and (60), we get

$$\begin{aligned}\mathrm{M}_k^{(x)} &\triangleq \left[\left(\mathbf{G}_k^H \mathbf{R}_k \mathbf{G}_k\right)^{-1} \mathbf{G}_k^H \frac{\partial \mathbf{R}_k}{\partial x} \mathbf{G}_k\right]^2 \\ &= \frac{1}{\sigma^4} \sum_{\ell,j=1}^{L} \left[\mathbf{U}_\ell \otimes \mathbf{W}_{\ell,k}^{(x)} + \mathbf{O}_\ell \otimes \mathbf{Z}\mathbf{W}_{\ell,k}^{(x)}\right] \\ &\quad \cdot \left[\mathbf{U}_j \otimes \mathbf{W}_{j,k}^{(x)} + \mathbf{O}_j \otimes \mathbf{Z}\mathbf{W}_{j,k}^{(x)}\right] \\ &= \frac{1}{\sigma^4} \sum_{\ell,j=1}^{L} \mathbf{U}_\ell \mathbf{U}_j \otimes \mathbf{W}_{\ell,k}^{(x)} \mathbf{W}_{j,k}^{(x)} \\ &\quad + \mathbf{U}_\ell \mathbf{O}_j \otimes \mathbf{W}_{\ell,k}^{(x)} \mathbf{Z}\mathbf{W}_{j,k}^{(x)} \\ &\quad + \mathbf{O}_\ell \mathbf{U}_j \otimes \mathbf{Z}\mathbf{W}_{\ell,k}^{(x)} \mathbf{W}_{j,k}^{(x)} \\ &\quad + \mathbf{O}_\ell \mathbf{O}_j \otimes \mathbf{Z}\mathbf{W}_{\ell,k}^{(x)} \mathbf{Z}\mathbf{W}_{j,k}^{(x)}.\end{aligned} \tag{61}$$

Note that

$$\mathrm{tr}\{\mathbf{U}_\ell \mathbf{U}_j\} = 2(1 - L\delta_{\ell,j}) \tag{62}$$
$$\mathrm{tr}\{\mathbf{U}_\ell \mathbf{O}_j\} = \mathrm{tr}\{\mathbf{O}_\ell \mathbf{U}_j\} = -L(1 - L\delta_{\ell,j}) \tag{63}$$
$$\mathrm{tr}\{\mathbf{O}_\ell \mathbf{O}_j\} = 0. \tag{64}$$

Thus, the trace of (61) is

$$\begin{aligned}\mathrm{tr}\left\{\mathrm{M}_k^{(x)}\right\} &= \sigma^{-4} \sum_{\ell,j=1}^{L} \mathrm{tr}\{\mathbf{U}_\ell \mathbf{U}_j\} \mathrm{tr}\left\{\mathbf{W}_{\ell,k}^{(x)} \mathbf{W}_{j,k}^{(x)}\right\} \\ &\quad + \mathrm{tr}\{\mathbf{U}_\ell \mathbf{O}_j\} \mathrm{tr}\left\{\mathbf{W}_{\ell,k}^{(x)} \mathbf{Z}\mathbf{W}_{j,k}^{(x)}\right\} \\ &\quad + \mathrm{tr}\{\mathbf{O}_\ell \mathbf{U}_j\} \mathrm{tr}\left\{\mathbf{Z}\mathbf{W}_{\ell,k}^{(x)} \mathbf{W}_{j,k}^{(x)}\right\} \\ &\quad + \mathrm{tr}\{\mathbf{O}_\ell \mathbf{O}_j\} \mathrm{tr}\left\{\mathbf{Z}\mathbf{W}_{\ell,k}^{(x)} \mathbf{Z}\mathbf{W}_{j,k}^{(x)}\right\} \\ &= \sigma^{-4} \sum_{\ell,j=1}^{L} \mathrm{tr}\{\mathbf{U}_\ell \mathbf{U}_j\} \mathrm{tr}\left\{\mathbf{W}_{\ell,k}^{(x)} \mathbf{W}_{j,k}^{(x)}\right\} \\ &\quad + \mathrm{tr}\{\mathbf{U}_\ell \mathbf{O}_j\} \left[\mathrm{tr}\left\{\mathbf{W}_{\ell,k}^{(x)} \mathbf{Z}\mathbf{W}_{j,k}^{(x)}\right\}\right. \\ &\quad \left. + \mathrm{tr}\left\{\mathbf{Z}\mathbf{W}_{\ell,k}^{(x)} \mathbf{W}_{j,k}^{(x)}\right\}\right].\end{aligned} \tag{65}$$

Note that

$$\begin{aligned}&\mathrm{tr}\left\{\mathbf{W}_{\ell,k}^{(x)} \mathbf{W}_{j,k}^{(x)}\right\} \\ &= \mathrm{tr}\left\{\left[\dot{\mathbf{P}}\dot{\tau}_{\ell,k}^{(x)} - i2\pi\tilde{\mathbf{T}}_{\ell,k}\mathbf{P}\dot{f}_{\ell,k}^{(x)}\right]\right. \\ &\quad \left. \times \left[\dot{\mathbf{P}}\dot{\tau}_{j,k}^{(x)} - i2\pi\tilde{\mathbf{T}}_{j,k}\mathbf{P}\dot{f}_{j,k}^{(x)}\right]\right\} \\ &= \dot{\tau}_{\ell,k}^{(x)}\dot{\tau}_{j,k}^{(x)} \mathrm{tr}\{\dot{\mathbf{P}}^2\} - (2\pi)^2 \dot{f}_{\ell,k}^{(x)} \dot{f}_{j,k}^{(x)} \mathrm{tr}\{\tilde{\mathbf{T}}_{\ell,k}\mathbf{P}\tilde{\mathbf{T}}_{j,k}\mathbf{P}\}.\end{aligned} \tag{66}$$

Observe that $\mathbf{P}, \dot{\mathbf{P}}$ are positive-definite Toeplitz matrices, and infinite dimensional matrices of this type preserve their Toeplitz structure under matrix multiplication, addition and inversion. Further $\mathrm{tr}\{\mathbf{T}\} = 0$ because the time interval is symmetric around zero. Thus,

$$\begin{aligned}\mathrm{tr}\left\{\tilde{\mathbf{T}}_{\ell,k}\mathbf{P}\dot{\mathbf{P}}\right\} &= \mathrm{tr}\left\{\mathbf{F}_{\ell,k}^H \mathbf{T} \mathbf{F}_{\ell,k} \mathbf{P}\dot{\mathbf{P}}\right\} \\ &= \mathrm{tr}\left\{\mathbf{T}\mathbf{F}_{\ell,k}\mathbf{P}\dot{\mathbf{P}}\mathbf{F}_{\ell,k}^H\right\} = 0.\end{aligned} \tag{67}$$

The last term in (65) is

$$\begin{aligned}&\mathrm{tr}\left\{\mathbf{Z}\mathbf{W}_{\ell,k}^{(x)} \mathbf{W}_{j,k}^{(x)}\right\} \\ &= \mathrm{tr}\left\{\mathbf{Z}\left[\dot{\mathbf{P}}\dot{\tau}_{\ell,k}^{(x)} - i2\pi\tilde{\mathbf{T}}_{\ell,k}\mathbf{P}\dot{f}_{\ell,k}^{(x)}\right]\right. \\ &\quad \left. \times \left[\dot{\mathbf{P}}\dot{\tau}_{j,k}^{(x)} - i2\pi\tilde{\mathbf{T}}_{j,k}\mathbf{P}\dot{f}_{j,k}^{(x)}\right]\right\} \\ &= \dot{\tau}_{\ell,k}^{(x)}\dot{\tau}_{j,k}^{(x)} \mathrm{tr}\{\mathbf{Z}\dot{\mathbf{P}}^2\} - (2\pi)^2 \dot{f}_{\ell,k}^{(x)} \dot{f}_{j,k}^{(x)} \mathrm{tr}\{\mathbf{Z}\tilde{\mathbf{T}}_{\ell,k}\mathbf{P}\tilde{\mathbf{T}}_{j,k}\mathbf{P}\}.\end{aligned} \tag{68}$$

Now (65) becomes

$$\begin{aligned}&\mathrm{tr}\left\{\mathrm{M}_k^{(x)}\right\} \\ &= \sigma^{-4} \sum_{\ell,j=1}^{L} 2(1 - L\delta_{\ell,j}) \left(\dot{\tau}_{\ell,k}^{(x)}\dot{\tau}_{j,k}^{(x)} \left[\mathrm{tr}\{\dot{\mathbf{P}}^2\} - L\mathrm{tr}\{\mathbf{Z}\dot{\mathbf{P}}^2\}\right]\right. \\ &\quad \left. -(2\pi)^2 \dot{f}_{\ell,k}^{(x)} \dot{f}_{j,k}^{(x)} \frac{\mathrm{tr}\{\tilde{\mathbf{T}}_{\ell,k}\tilde{\mathbf{T}}_{j,k}\}}{N} \left[\mathrm{tr}\{\mathbf{P}^2\} - L\mathrm{tr}\{\mathbf{Z}\mathbf{P}^2\}\right]\right).\end{aligned} \tag{69}$$

Note that the terms in (69) satisfy the relations

$$\begin{aligned}\mathbf{P}^2 - L\mathbf{P}^2\mathbf{Z} &\simeq \frac{\sigma^2}{L}\mathbf{P}, \\ \dot{\mathbf{P}}^2 - L\dot{\mathbf{P}}^2\mathbf{Z} &\simeq \frac{\sigma^2}{L}\dot{\mathbf{P}}^2\mathbf{P}^{-1}, \\ \mathrm{tr}\{\tilde{\mathbf{T}}_{\ell,k}\tilde{\mathbf{T}}_{j,k}\} &= \sum_{n=1}^{N} (t_n - \tau_{\ell,k})(t_n - \tau_{j,k}) \\ &\simeq \frac{NT^2}{12}.\end{aligned} \tag{70}$$

Using (70) in (69) yields

$$\begin{aligned}&\mathrm{tr}\left\{\mathrm{M}_k^{(x)}\right\} \\ &= \frac{2}{L\sigma^2} \mathrm{tr}\{\dot{\mathbf{P}}^2\mathbf{P}^{-1}\} \sum_{\ell,j=1}^{L} (1 - L\delta_{\ell,j}) \dot{\tau}_{\ell,k}^{(x)}\dot{\tau}_{j,k}^{(x)} \\ &\quad - \frac{2(2\pi)^2}{NL\sigma^2} \mathrm{tr}\{\mathbf{P}\} \sum_{\ell,j=1}^{L} (1 - L\delta_{\ell,j}) \dot{f}_{\ell,k}^{(x)} \dot{f}_{j,k}^{(x)} \mathrm{tr}\{\tilde{\mathbf{T}}_{\ell,k}\tilde{\mathbf{T}}_{j,k}\} \\ &= \frac{2}{L\sigma^2} \mathrm{tr}\{\dot{\mathbf{P}}^2\mathbf{P}^{-1}\} \sum_{\ell,j=1}^{L} (1 - L\delta_{\ell,j}) \dot{\tau}_{\ell,k}^{(x)}\dot{\tau}_{j,k}^{(x)} \\ &\quad - \frac{2\pi^2 NT^2}{3L\sigma^2} R_{ss}(0) \sum_{\ell,j=1}^{L} (1 - L\delta_{\ell,j}) \dot{f}_{\ell,k}^{(x)} \dot{f}_{j,k}^{(x)}.\end{aligned} \tag{71}$$

The sums in (71) have the following compact representations:

$$\sum_{\ell,j=1}^{L}(1-L\delta_{\ell,j})\dot{\tau}_{\ell,k}^{(x)}\dot{\tau}_{j,k}^{(x)} = -L\left[\dot{\boldsymbol{\tau}}_k^{(x)}\right]^T \mathbf{B}\dot{\boldsymbol{\tau}}_k^{(x)},$$
$$\sum_{\ell,j=1}^{L}(1-L\delta_{\ell,j})\dot{f}_{\ell,k}^{(x)}\dot{f}_{j,k}^{(x)} = -L\left[\dot{\mathbf{f}}_k^{(x)}\right]^T \mathbf{B}\dot{\mathbf{f}}_k^{(x)} \quad (72)$$

where we used the definitions

$$\mathbf{B} \triangleq \mathbf{I}_L - (1/L)\mathbf{1}_{L,L}$$
$$\dot{\mathbf{f}}_k^{(x)} \triangleq \left[\dot{f}_{1,k}^{(x)}, \dots \dot{f}_{L,k}^{(x)}\right]^T$$
$$\dot{\boldsymbol{\tau}}_k^{(x)} \triangleq \left[\dot{\tau}_{1,k}^{(x)}, \dots \dot{\tau}_{L,k}^{(x)}\right]^T. \quad (73)$$

Consider a Toeplitz Hermitian matrix $[\bar{\mathbf{T}}]_{i,j} = \bar{t}_{i-j}$ and the discrete time Fourier transform (DTFT)

$$\mu(\omega) \triangleq \sum_{n=-\infty}^{\infty} \bar{t}_n e^{-j\omega n}. \quad (74)$$

Define the unitary matrix

$$[\mathbf{U}]_{i,k} \triangleq N^{-1/2} e^{-j2\pi(k-1)(i-1)/N} \quad (75)$$

associated with DFT, the vector $\mathbf{d}$ given by

$$[\mathbf{d}]_k \triangleq \mu(\omega_k); \quad \omega_k \triangleq 2\pi(k-1)/N, \quad (76)$$

the diagonal matrix $\mathbf{D} \triangleq \mathrm{diag}\{\mathbf{d}\}$ and the circulant matrix $\mathbf{C} \triangleq \mathbf{U}^H\mathbf{D}\mathbf{U}$. It can be shown that under mild regularity conditions $\bar{\mathbf{T}}$ converges to $\mathbf{C}$ as N increases [18], [19]. Note that, for large N, we have $\mathrm{tr}\{\bar{\mathbf{T}}\} = \mathrm{tr}\{\mathbf{C}\} = \mathrm{tr}\{\mathbf{D}\}$ and also if we have several matrices $\bar{\mathbf{T}}_i = \mathbf{C}_i = \mathbf{U}^H\mathbf{D}_i\mathbf{U}$ then $\mathrm{tr}\{\bar{\mathbf{T}}_1^2\bar{\mathbf{T}}_2\} = \mathrm{tr}\{\mathbf{D}_1^2\mathbf{D}_2\}$. Using this result, and replacing $\bar{\mathbf{T}}_1$ with $\dot{\mathbf{P}}$ and $\bar{\mathbf{T}}_2$ with $\mathbf{P}^{-1}$, we have

$$\mathrm{tr}\{\dot{\mathbf{P}}^2\mathbf{P}^{-1}\} = \mathrm{tr}\{\dot{\mathbf{D}}^2\mathbf{D}^{-1}\} \quad (77)$$

where

$$\mathbf{D}_{k,k} = \sum_{n=-\infty}^{\infty} R(nT_s)e^{-j\omega_k n} \triangleq S(\omega_k)$$
$$\dot{\mathbf{D}}_{k,k} = \sum_{n=-\infty}^{\infty} \dot{R}(nT_s)e^{-j\omega_k n} = j(\omega_k/T_s)S(\omega_k). \quad (78)$$

We get

$$\mathrm{tr}\{\dot{\mathbf{P}}^2\mathbf{P}^{-1}\} = -\frac{N}{2\pi T_s^2}\int_{-\pi}^{\pi}\omega^2 S(\omega)\,d\omega = -\frac{N}{2\pi}\int_{-\infty}^{\infty}\Omega^2\tilde{S}(\Omega)\,d\Omega \quad (79)$$

where Ω is the frequency in radian per second and $\tilde{S}(\Omega)$ is the signal spectrum. If the noise spectral density is N_0 [W/rad/s] we have $\sigma^2 = N_0/Ts = N_0N/T$. Now (71) becomes

$$\mathrm{tr}\left\{\mathrm{M}_k^{(x)}\right\} = \frac{T}{N_0\pi}\left[\dot{\boldsymbol{\tau}}_k^{(x)}\right]^T \mathbf{B}\dot{\boldsymbol{\tau}}_k^{(x)}\int_{-\infty}^{\infty}\Omega^2\tilde{S}(\Omega)\,d\Omega$$
$$+ \frac{\pi T^3}{3N_0}\left[\dot{\mathbf{f}}_k^{(x)}\right]^T \mathbf{B}\dot{\mathbf{f}}_k^{(x)}\int_{-\infty}^{\infty}\tilde{S}(\Omega)\,d\Omega$$
$$= \frac{(WT)\mathrm{SNR}}{12\pi}\left(W^2\left[\dot{\boldsymbol{\tau}}_k^{(x)}\right]^T \mathbf{B}\dot{\boldsymbol{\tau}}_k^{(x)} + (2\pi T)^2\left[\dot{\mathbf{f}}_k^{(x)}\right]^T \mathbf{B}\dot{\mathbf{f}}_k^{(x)}\right) \quad (80)$$

where $\mathrm{SNR} \triangleq \tilde{S}(\Omega)/N_0$ is assumed constant over the interval $[-W/2, W/2]$ [rad/s] and zero outside this interval. Define now

$$\mathbf{V}_f^{(x)} \triangleq \mathbf{B}\left[\dot{\mathbf{f}}_1^{(x)}, \dots \dot{\mathbf{f}}_K^{(x)}\right]$$
$$\mathbf{V}_f^{(y)} \triangleq \mathbf{B}\left[\dot{\mathbf{f}}_1^{(y)}, \dots \dot{\mathbf{f}}_K^{(y)}\right]$$
$$\mathbf{V}_\tau^{(x)} \triangleq \mathbf{B}\left[\dot{\boldsymbol{\tau}}_1^{(x)}, \dots \dot{\boldsymbol{\tau}}_K^{(x)}\right]$$
$$\mathbf{V}_\tau^{(y)} \triangleq \mathbf{B}\left[\dot{\boldsymbol{\tau}}_1^{(y)}, \dots \dot{\boldsymbol{\tau}}_K^{(y)}\right]. \quad (81)$$

To complete the analysis we provide expressions for the derivatives of the delay and the Doppler w.r.t. x and y. From (2) and (3), we get

$$(c/f_c)f_{\ell,k} = \frac{v_{\ell,k}^x(x-x_{\ell,k}) + v_{\ell,k}^y(y-y_{\ell,k})}{\sqrt{(x-x_{\ell,k})^2+(y-y_{\ell,k})^2}} \quad (82)$$

and therefore

$$(c/f_c)\frac{\partial f_{\ell,k}}{\partial x}$$
$$= \frac{v_{\ell,k}^x\left[(x-x_{\ell,k})^2+(y-y_{\ell,k})^2\right]}{\left[(x-x_{\ell,k})^2+(y-y_{\ell,k})^2\right]^{3/2}} - \frac{\left[v_{\ell,k}^x(x-x_{\ell,k})+v_{\ell,k}^y(y-y_{\ell,k})\right](x-x_{\ell,k})}{\left[(x-x_{\ell,k})^2+(y-y_{\ell,k})^2\right]^{3/2}}$$
$$= \frac{v_{\ell,k}^x}{\sqrt{(x-x_{\ell,k})^2+(y-y_{\ell,k})^2}} - \frac{\left[v_{\ell,k}^x(x-x_{\ell,k})+v_{\ell,k}^y(y-y_{\ell,k})\right](x-x_{\ell,k})}{\left[(x-x_{\ell,k})^2+(y-y_{\ell,k})^2\right]^{3/2}}$$
$$= \frac{v_{\ell,k}^x}{||\Delta\mathbf{p}_{\ell,k}||} - \frac{\mathbf{v}_{\ell,k}^T\Delta\mathbf{p}_{\ell,k}(x-x_{\ell,k})}{||\Delta\mathbf{p}_{\ell,k}||^3}$$
$$= \frac{v_{\ell,k}^x}{d_{\ell,k}} - \frac{||\mathbf{v}_{\ell,k}||}{d_{\ell,k}}\cos\phi_{\ell,k}\cos\theta_{\ell,k}$$
$$= \frac{||\mathbf{v}_{\ell,k}||}{d_{\ell,k}}\sin\phi_{\ell,k}\sin\theta_{\ell,k}. \quad (83)$$

Similarly,

$$(c/f_c)\frac{\partial f_{\ell,k}}{\partial y} = \frac{v^y_{\ell,k}}{\|\Delta \mathbf{p}_{\ell,k}\|} - \frac{\mathbf{v}^T_{\ell,k}\Delta \mathbf{p}_{\ell,k}(y - y_{\ell,k})}{\|\Delta \mathbf{p}_{\ell,k}\|^3}$$
$$= \frac{v^y_{\ell,k}}{d_{\ell,k}} - \frac{\|\mathbf{v}_{\ell,k}\|}{d_{\ell,k}}\cos\phi_{\ell,k}\sin\theta_{\ell,k}$$
$$= -\frac{\|\mathbf{v}_{\ell,k}\|}{d_{\ell,k}}\sin\phi_{\ell,k}\cos\theta_{\ell,k}. \quad (84)$$

Here, ϕ is the angle between the platform velocity vector and the line connecting the platform and the emitter. The angle θ is the angle between the line connecting the platform and the emitter and the x axis. We use $d_{\ell,k}$ to denote the distance between the emitter and the platform.

Also note that

$$c\tau_{\ell,k} = \|\mathbf{p} - \mathbf{p}_{\ell,k}\|$$
$$c\frac{\partial \tau_{\ell,k}}{\partial x} = \frac{x - x_{\ell,k}}{\|\mathbf{p} - \mathbf{p}_{\ell,k}\|} = \cos\theta_{\ell,k}$$
$$c\frac{\partial \tau_{\ell,k}}{\partial y} = \frac{y - y_{\ell,k}}{\|\mathbf{p} - \mathbf{p}_{\ell,k}\|} = \sin\theta_{\ell,k}. \quad (85)$$

Finally, recalling that

$$\mathbf{J}_{1,1} = \sum_{k=1}^{K} \mathrm{tr}\left\{\mathrm{M}^{(x)}_k\right\}$$
$$\mathbf{J}_{2,2} = \sum_{k=1}^{K} \mathrm{tr}\left\{\mathrm{M}^{(y)}_k\right\}$$
$$\mathbf{J}_{1,2} = \sum_{k=1}^{K} \mathrm{tr}\left\{\mathrm{M}^{(x,y)}_k\right\} \quad (86)$$

we get from (80) the result in (30).

This concludes the derivation.

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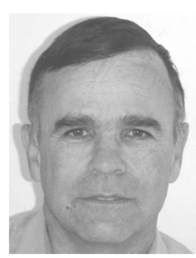

Anthony J. Weiss (S'84–M'85–SM'86–F'97) received the B.Sc. degree from the Technion-Israel Institute of Technology, Haifa, in 1973 and the M.Sc. and Ph.D. degrees from Tel Aviv University, Tel Aviv, Israel, in 1982 and 1985, all in electrical engineering.

From 1973 to 1983, he was involved in research and development of numerous projects in the fields of communications, command and control, and emitter localization. In 1985, he joined the Department of Electrical Engineering-Systems, Tel Aviv University. From 1996 to 1999, he served as the department chairman. He is currently the head of the Electrical Engineering School in Tel Aviv University. His research interests include detection and estimation theory, signal processing, sensor array processing, and wireless networks. He also held leading scientific positions with Signal Processing Technology, Ltd., Wireless on Line, Ltd., and SigmaOne, Ltd. He published over 100 papers in professional magazines and conferences and holds nine U.S. patents.

Prof. Weiss was a recipient of the IEEE 1983 Acoustics, Speech, and Signal Processing Society's Senior Award and the IEEE third millennium medal. He served as the Chairman of the IEEE Israel Section.