Accurate Discretization of Analog Audio Filters with Application to Parametric Equalizer Design

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***Abstract*—This article is concerned with accurate discretization of linear analog filters such that the frequency response of the discrete time filter accurately matches that of the continuous time filter. The approach is based on formal reconstruction of the continuous time signal using Shannon's interpolation theorem and numerical solving of the differential equation corresponding to the analog filter. When the formal continuous time system is sampled, the resulting filter reduces to discrete linear filter, which can be realized either as a state space model or as an IIR filter. The proposed methodology is applied to design of filters for parametric equalizers.**

***Index Terms*—analog filter, discretization, Shannon's interpolation, differential equation, parametric equalizer**

I. Introduction

THE need for accurate discretization of analog filters arises, for example, in the design of virtual analog synthesizers and modeling of audio effects [1], [2], [3], [4], [5]. Parametric equalizers and other linear filters are used as sub-blocks of such systems and therefore their accurate discretization is also important – and surprisingly non-trivial. The brute-force approach would be to design a high order FIR filter with standard design methods (see, e.g., [6], [7], [8]), but unfortunately, the resulting filter is computationally heavy and has a long delay. For this reason IIR filter design methods are more often used.

A common method of designing IIR filters for audio applications is the bilinear transformation, which has the problem that it causes significant distortion of the frequency and phase responses at high frequencies, near the Nyquist frequency [9]. One method of coping with this problem is to modify the bilinear transformation a bit such that the magnitude response at the Nyquist frequency is closer to that of the original analog system [9], [10]. Another possible approach is to use numerical optimization [11], [12], which is computationally heavy procedure and thus cannot be used in applications, where the coefficients need to be computed *in situ* (as in audio equalizers). In this article we shall approach the problem from a bit different perspective, that is, by modeling and approximating the sampling and filtering operations directly using ordinary differential equation (ODE) methods from control theory [13], [14]. This results in a method that is computationally light, but still produces better approximations than the bilinear transformation based methods.

In this article, we shall present a general methodology that can be used for discretizing analog filters such that the frequency response of the digital filter accurately matches that of the analog filter. The methodology can be used with systems where the continuous input signal can be assumed to be band-limited to the frequency range from zero to Nyquist, which applies especially well to audio signal processing applications. In principle, the methodology can be used in any application instead of the traditional discretization methods such as impulse invariant transformation and bilinear z-transformation (see, e.g., [6], [7], [8]).

The proposed method is based on quite elementary ideas, namely to approximation of the Shannon interpolation and numerical solutions of the corresponding LTI differential equations. Thus, it is possible that similar methodology has been already used, for example, in design of commercial digital equalizer systems. However, to the authors' knowledge, such general methodology has not been published before.

The advantage of the proposed methodology over the bilinear transformation and related methods is that the frequency response can be made accurate also near the Nyquist frequency, which is hard with the traditional discretization methods. The disadvantage of the methods is that it introduces a slight delay to the signal (order of tens of samples), which limits its applicability to control engineering applications. However, the delay is a few orders of magnitude shorter than with FIR based filter design methods.

A. Idea of Method

The problem of accurate approximation of the analog system response is re-casted into problem of:

1) Estimating the real input signal from discrete measurements.
2) Solving the analog system response with the real input signal.

Because the input signal is band-limited, the first of these two problems is solved by help of the Shannon's interpolation theorem [15]. The second problem is solved using the methods from theory of linear time invariant differential equations, that is, linear state space models borrowed from control theory [13], [14].

The step 1 above can be interpreted to interpolate all the possible values between the discrete samples and thus it can be considered as oversampling to infinite sampling frequency (i.e., to continuous time). The step 2 approximates the analog filter for the infinitely oversampled signal, which

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as a continuous-time operation means solving the differential equation corresponding to the analog filter.

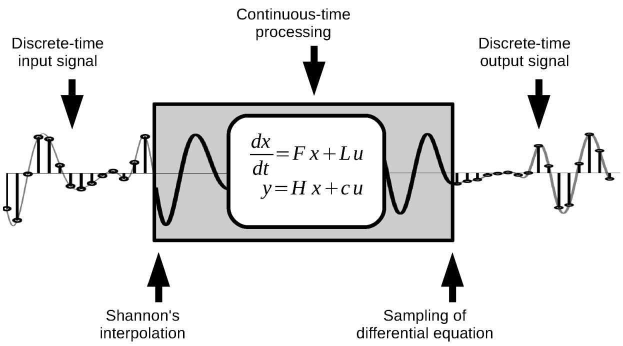


Fig. 1. The idea of the approach is to first reconstruct the continuous time signal using Shannon's interpolation theorem, then formally apply the filter to the continuous-time signal and finally sample the output.

As shown in Figure 1 the proposed approach can be interpreted such that the sampled input signal is first converted into continuous-time signal. Of course, the continuous time signal cannot be realized in a computer, but we can compute the parameters of the formula that would reproduce the signal accurately. This continuous-time signal is then fed into the continuous-time filter. The continuous-time filtering operation can be mathematically represented as solving a certain linear time invariant differential equation (or state space model). The exact response of the continuous-time filter can be now computed by formally solving the differential equation with the input reconstruction formula as the input function. By sampling the differential equation solution we get the same result as if really did the filtering in continuous time and sampled the continuous-time system response at discrete times.

As a practical application of the methodology we shall show how the methods can be used for designing a discrete parametric equalizer, whose magnitude and phase responses match the analog counterparts accurately.

B. Discretization of Analog Audio Equalizers

The design of digital time domain parametric audio equalizers is often based on discretizing an analog parametric equalizer, which is composed of chain of peaking and shelving filters [8]. Although, in principle, we could skip the analog domain design altogether and design discrete equalizer directly, for historical reasons and due to the superior audio quality of analog designs, this approach is still commonly used. In this article, we shall present a numerical approach that can be used for constructing accurate, but computationally light discrete approximations to these analog equalizers.

The discretization of the analog peaking and shelving filters is typically done using some of the well known s to z domain mapping based discretization methods, which are well documented in various books on digital signal processing (see, e.g., [6], [7], [8]). Due to its simplicity the bilinear transformation still seems to be a common choice nowadays.

All of the standard discretizations have their own weaknesses. For example, the problem in bilinear transformation is that it maps the whole analog frequency range $[0, \infty)$ onto discrete frequencies from 0 to Nyquist $[0, f_s/2)$, where f_s is the sampling frequency. The disadvantage of this is that due to the non-linear mapping of frequencies, only part of the frequency responses of analog and digital filters can be made to match. In particular, the behavior near the Nyquist frequency is problematic in digital equalizers.

Orfanidis [9] has suggested an improvement to the bilinear transformation based filters, where the gain of the discrete filter is matched to the gain of the analog filter at Nyquist frequency. This results in different behavior near the Nyquist frequency, because normally the gain of bilinear transformation based filter is zero at Nyquist frequency. However, the disadvantage of Orfanidis' discretization is that it only attempts to approximate the magnitude response at certain prescribed points and thus the behavior outside these points can still be anything. Also the phase response of the filter is not accurately matched to the analog filter.

The common way to discretize systems in control theory [13], [14] is to approximate the input using, for example, zeroth order hold (ZOH) or first order hold (FOH). These can result in quite good approximations of the analog filter, but because nor the piecewise constant or piecewise linear approximation works well with high frequencies, they are problematic near the Nyquist frequency. For this reason, in this article we shall apply Shannon's interpolation theorem to input reconstruction, which even in approximate setting results in much better approximations to the analog filter at high frequencies.

Yet another way to design filters would be to design a FIR filter with a desired frequency response using the well known methods such as window method (see, e.g., [6], [7], [8]). The difficulty with FIR filters is that in order to get the response at low frequencies right, we would need extensively long filters. This would result in couple of orders of magnitude more computations per sample when compared to the IIR designs presented here.

C. Analog Equalizer Filter Prototypes

The equalizers analyzed in this article will be based on the following common filter prototypes. Each of the filters can be written either as non-proper transfer functions or as a sum of constant and a proper transfer function

- *Peaking* filter can be used for amplifying or attenuating certain narrow frequency band. In audio terms, it can be used for modifying the loudness of certain middle range of frequencies. The Laplace domain transfer function can be written as

$$\begin{aligned} G_P(s) &= \frac{s^2 + (K\omega_0/Q)\,s + \omega_0^2}{s^2 + \omega_0/Q/K\,s + \omega_0^2} \\ &= 1 + \frac{([K - 1/K]\omega_0/Q)\,s}{s^2 + (\omega_0/Q/K)\,s + \omega_0^2}. \end{aligned} \tag{1}$$

- *Low shelf* filter can be used for amplifying/attenuating the frequencies below certain frequency value, that is, modifying the loudness of bass in audio signal. The

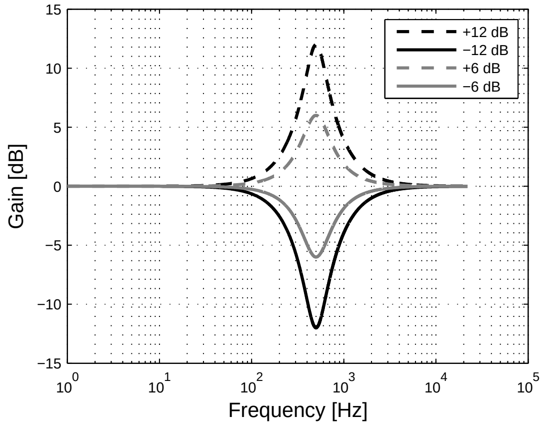


Fig. 2. Example of peaking filter responses with center frequency 500Hz and $Q = 1$.

transfer function can be written as:

$$\begin{aligned} G_L(s) &= \frac{s^2 + (\sqrt{K}\omega_0/Q)\, s + K\,\omega_0^2}{s^2 + (\omega_0/\sqrt{K}/Q)\, s + \omega_0^2/K} \\ &= 1 + \frac{([\sqrt{K} - 1/\sqrt{K}]\omega_0/Q)\, s + (K - 1/K)\,\omega_0^2}{s^2 + (\omega_0/\sqrt{(K)}/Q)\, s + \omega_0^2/K}. \end{aligned} \tag{2}$$

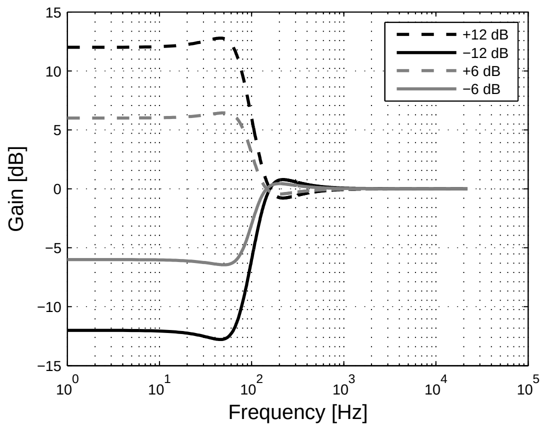


Fig. 3. Example of low shelf filter responses with center frequency 100Hz and $Q = 1$.

- *High shelf* filter can be used for amplifying/attenuating the frequencies above certain frequency value, that is, modifying the loudness of treble in audio signal. The transfer function can be written as:

$$\begin{aligned} G_H(s) &= \frac{K^2\, s^2 + \sqrt{K}\omega_0/Q\, s + K\,\omega_0^2}{s^2 + \sqrt{K}\omega_0/Q\, s + K\,\omega_0^2} \\ &= K^2 + \frac{([\sqrt{K} - K^2\,\sqrt{K}]\omega_0/Q\, s + (K - K^3)\,\omega_0^2}{s^2 + \sqrt{K}\omega_0/Q\, s + K\,\omega_0^2}. \end{aligned} \tag{3}$$

In all of the above filters the positive constants K, Q and ω_0 have been defined as follows:

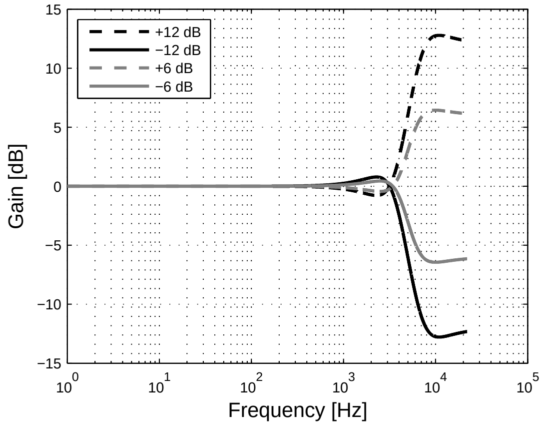


Fig. 4. Example of high shelf filter responses with center frequency 5000Hz and $Q = 1$.

- K defines the gain of the filter at the amplified/attenuated band such that if g_{db} is the gain in decibels, we have
$$K = 10^{g_{db}/40}. \tag{4}$$
- Q defines the quality or the steepness of the transition from neutral to amplified/attenuated band.
- ω_0 is the active angular velocity, that is, the position of the peak in peaking filter and the position of the transition in low and high shelf filters. The angular velocity is related to the corresponding frequency f_0 in the following simple manner:
$$\omega_0 = 2\pi f_0. \tag{5}$$

II. MAIN RESULTS

A. State Space Discretization of Analog Filter

The Laplace domain transfer function of a generic analog filter can be written in the following form:

$$G(s) = c + \frac{b_1\, s^{m-1} + \cdots + b_{m-1}\, s + b_m}{s^m + a_1\, s^{m-1} + \cdots + a_{m-1}\, s + a_m}, \tag{6}$$

where $c, a_1 \ldots a_m, b_1 \ldots b_m$ are some known constants. The transfer function consists of two parts: first one is a feed-through part, which simply multiplies the input signal with constant c and the second part is a proper transfer function and thus realizable as a state space model. The state space model can be realized, for example, utilizing the observer canonical form [14], which results in the following:

$$\begin{aligned} \frac{d\mathbf{x}(t)}{dt} &= \underbrace{\begin{pmatrix} -a_1 & 1 & & & \\ -a_2 & & 1 & & \\ \vdots & & & \ddots & \\ -a_{m-1} & & & & 1 \\ -a_m & 0 & 0 & \cdots & 0 \end{pmatrix}}_{\mathbf{F}} \mathbf{x}(t) + \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{m-1} \\ b_m \end{pmatrix}}_{\mathbf{L}} u(t) \\ y(t) &= \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}}_{\mathbf{H}} \mathbf{x}(t) + c\, u(t). \end{aligned} \tag{7}$$

With given initial state $\mathbf{x}(0)$ the explicit solution to the differential equation of the state can be now written as

$$\mathbf{x}(t) = e^{t\,\mathbf{F}}\,\mathbf{x}(0) + \int_0^t e^{(t-s)\,\mathbf{F}}\,\mathbf{L}\,u(t)\,ds, \tag{8}$$

where $e^{t\,\mathbf{F}}$ is the matrix exponential of $t\,\mathbf{F}$, defined as

$$e^{t\,\mathbf{F}} = \mathbf{I} + t\,\mathbf{F} + \frac{1}{2!}\,t^2\,\mathbf{F}^2 + \frac{1}{3!}\,t^3\,\mathbf{F}^3 + \dots \tag{9}$$

The output of the system is then $y(t) = x_1(t) + c\,u(t)$.

In discrete time filtering, we are only interested in the values of state and output on discrete instants of time $t_0, t_1, t_2, \ldots$, where $\Delta t = t_k - t_{k-1}$ is the sampling period. In that case it is useful to write the solution in the following recursive form:

$$\begin{aligned} \mathbf{x}_k &= e^{\Delta t\,\mathbf{F}}\,\mathbf{x}_{k-1} + \int_0^{\Delta t} e^{(\Delta t - s)\,\mathbf{F}}\,\mathbf{L}\,u(t_{k-1}+s)\,ds \\ y_k &= \mathbf{H}\,\mathbf{x}_k + c\,u_k, \end{aligned} \tag{10}$$

where $\mathbf{x}_k = \mathbf{x}(t_k)$, $y_k = y(t_k)$ and $u_k = u(t_k)$. With these equations, given the previous state value $\mathbf{x}_{k-1}$ we can *exactly* compute the next state value $\mathbf{x}_k$ and thus also exactly reconstruct outputs at the discrete time instants. However, in order to do this, we will need to know the input values also between the discrete time instants, but we shall return to this issue in the next section. Note that with the recursive definition (10) we no longer need to assume the existence of certain start time and initial condition $\mathbf{x}(0)$, but instead, we can freely choose the time step indexing as $k = -\infty, \ldots, \infty$. This shall be assumed from now on.

The matrix exponential $e^{\Delta t\,\mathbf{F}}$ can be easily computed using various methods such as with various numerical software or analytically, for example, using the Taylor series expansion, Laplace transform or Cayley-Hamilton theorem [13].

B. Input Reconstruction by Shannon's Interpolation

In the previous section we derived the Equations (10), which can be used for solving the response of the analog system exactly. However, in order to compute the integral in the state equation, we should know the input signal also between the sample points. One way to proceed would be to approximate the input as piecewise constant or piecewise linear signal, which would result in ZOH or FOH discretizations [13]. However, this would only result in approximately correct discretization and there is no guarantee that naive increasing of approximation order would converge to the perfect reconstruction.

Fortunately, because the input signal is band-limited below the Nyquist frequency, it indeed is possible to reconstruct the original signal from its samples. This exact interpolation function is given by the *Shannon interpolation theorem* [15], which tells that a band-limited signal can be *exactly* reconstructed from its samples by the following sinc-interpolation formula:

$$u(t) = \sum_{j=-\infty}^{\infty} u_j\,\mathrm{sinc}\left(\frac{t - t_j}{\Delta t}\right), \tag{11}$$

where $\mathrm{sinc}(t) = \sin(\pi\,t)/(\pi\,t)$.

In practice, we need to truncate the series and analogously to the finite sampling rate case this will cause Gibbs phenomenon in spectral domain. In order to get rid of it, we need to apply a window function to the series. With window function $w_n(t)$ and $2n+1$ terms in the series around step $k-1$ results is:

$$u(t) = \sum_{j=-n}^{n} u_{k-j-1}\,\mathrm{sinc}\left(\frac{t - t_{k-j-1}}{\Delta t}\right)\,w_n(t - t_{k-j-1}). \tag{12}$$

A suitable window, which is utilized here is the Hamming window [16]:

$$w_n(t) = \begin{cases} 0.54 + 0.46\,\cos\left(\frac{\pi\,t}{n\,\Delta t}\right) & , \quad \text{for } |t| \le n\,\Delta t \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{13}$$

Thus the formula for the integral in Equations (10) is then given as

$$\int_0^{\Delta t} e^{(\Delta t - s)\,\mathbf{F}}\,\mathbf{L}\,u(t_{k-1}+s)\,ds = \sum_{j=-n}^{n} \mathbf{B}_j\,u_{k-j-1}, \tag{14}$$

where

$$\begin{aligned} \mathbf{B}_j = \int_0^{\Delta t} & e^{(\Delta t - s)\,\mathbf{F}}\,\mathbf{L}\,\mathrm{sinc}\left(\frac{s + j\,\Delta t}{\Delta t}\right) \\ & \times \left[0.54 + 0.46\cos\left(\frac{\pi\,[s + j\,\Delta t]}{n\,\Delta t}\right)\right]\,ds. \end{aligned} \tag{15}$$

These coefficients can be easily evaluated by using some suitable numerical integration scheme. In this article we shall use simple Simpson's rule, but more efficient methods could be developed by, for example, using the sinc part of the integral as the weight in Gaussian quadrature. In some cases the integral could even be evaluated in closed form or reduced to expression consisting of some standard special functions, which are easy to approximate numerically.

C. Implementing the Filter in State Space or IIR Form

Once the coefficients (15) have been computed, the system (10) can be implemented by delaying the output by $n\,\Delta t$ time units. The discretized model (10) reduces to form

$$\begin{aligned} \mathbf{x}_k &= \mathbf{A}\,\mathbf{x}_{k-1} + \sum_{j=0}^{2n} \mathbf{B}_{j-n}\,u_{k-j-1} \\ y_k &= \mathbf{H}\,\mathbf{x}_k + c\,u_{k-n}, \end{aligned} \tag{16}$$

where $\mathbf{A} = e^{\Delta t\,\mathbf{F}}$. This model can be implemented by using a simple delay line for the inputs. The corresponding z-transform domain IIR filter can be derived by taking the z-transform of the equations:

$$\begin{aligned} \mathbf{X}(z^{-1}) &= z^{-1}\mathbf{A}\,\mathbf{X}(z^{-1}) + \sum_{j=0}^{2n} \mathbf{B}_{j-n}\,z^{-j-1}\,U(z^{-1}) \\ Y(z^{-1}) &= \mathbf{H}\,\mathbf{X}(z^{-1}) + c\,z^{-n}U(z^{-1}). \end{aligned} \tag{17}$$

Solving for $Y(z^{-1})$ gives equation of the form $Y(z^{-1}) = G(z^{-1})\,U(z^{-1})$, where the transfer function is

$$G(z^{-1}) = \mathbf{H}\,(\mathbf{I} - z^{-1}\mathbf{A})^{-1}\left[\sum_{j=0}^{2n} \mathbf{B}_{j-n}\,z^{-j-1}\right] + c\,z^{-n}, \tag{18}$$

which is a IIR filter with numerator order $2n+m$ and denominator order m. The numerator order is the given because the adjoint of the matrix $(\mathbf{I} - z^{-1}\mathbf{A})$ is of order $m-1$ in z^{-1} and the highest order in the sum is $2n+1$. The denominator of the filter is the determinant of the matrix $\mathbf{I} - z^{-1}\mathbf{A}$, which is of the order m in z^{-1}. Note that the transfer function is purely causal such that the output at time step k only depends on inputs on time steps $k-1$ and before. If there is no need for such single-sample computation delay, the input can be delayed $n-1$ samples instead of n, which would drop the numerator order to $2n+m-1$.

III. Application to Parametric Equalizer Design

A. Discretization of Peaking and Shelving Filters

The peaking, low shelf and high shelf filters in Section I-C can be written in the following common form:

$$G(s) = c + \frac{b_1\,\omega_0\,s + b_2\,\omega_0^2}{s^2 + a_1\,\omega_0\,s + a_2\,\omega_0^2}, \tag{19}$$

where a_1, a_2, b_1, b_2, and c are simple functions of K and Q given in Section I-C. The transfer function can be realized, for example, as the state space model:

$$\begin{aligned} \frac{d\mathbf{x}(t)}{dt} &= \underbrace{\begin{pmatrix} -a_1\,\omega_0 & \omega_0 \\ -a_2\,\omega_0 & 0 \end{pmatrix}}_{\mathbf{F}} \mathbf{x}(t) + \underbrace{\begin{pmatrix} b_1\,\omega_0 \\ b_2\,\omega_0 \end{pmatrix}}_{\mathbf{L}} u(t) \\ y(t) &= \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{H}} \mathbf{x}(t) + c\,u(t), \end{aligned} \tag{20}$$

where $\mathbf{x}(t) = (x_1(t), x_2(t))$ is the state, $y(t)$ is the output and $u(t)$ is the input. Note that this is not exactly the observable canonical form, because we have rescaled x_2 to make the feedback matrix more well behaved.

In this case the matrix exponential $e^{\Delta t\,\mathbf{F}}$ can be evaluated explicitly. If the quality parameter $Q > 1/2$, we have

$$\mathbf{A} = e^{\Delta t\,\mathbf{F}} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{21}$$

where

$$\begin{aligned} d &= \sqrt{|4a_2 - a_1^2|} \\ A_{11} &= e^{-a_1\,\omega_0\,\Delta t/2}\,\cos(d\,\omega_0\,\Delta t/2) \\ &\quad - (a_1/d)e^{-a_1\,\omega_0\,\Delta t/2}\,\sin(d\,\omega_0\,\Delta t/2) \\ A_{12} &= (2/d)\,e^{-a_1\,\omega_0\,\Delta t/2}\,\sin(d\,\omega_0\,\Delta t/2) \\ A_{21} &= -(2a_2/d)\,e^{-a_1\,\omega_0\,\Delta t/2}\,\sin(d\,\omega_0\,\Delta t/2) \\ A_{22} &= e^{-a_1\,\omega_0 \Delta t/2}\,\cos(d\,\omega_0\,\Delta t/2) \\ &\quad + (a_1/d)e^{-a_1\,\omega_0\,\Delta t/2}\,\sin(d\,\omega_0\,\Delta t/2) \end{aligned} \tag{22}$$

If $Q < 1/2$, then the result is the same except that all $\sin$ and $\cos$ functions are changed to $\sinh$ and $\cosh$, respectively.

The expression for coefficients $\mathbf{B}_j$ in Equation (15) is now an integral, where the integrand consists of sinc functions, sines, cosines and exponential functions and thus is easy to evaluate numerically. The transfer function in Equation (18) now corresponds to an IIR with numerator order $2n+2$ and denominator order 2 in z^{-1}.

B. Cascading in Continuous vs. Discrete Time

We can now design equalizer by deriving a low shelf, a high self and a couple of peaking filters by using the procedure described in the previous section. The equalizer output will then be the result of applying each of the filters in cascade. The delay of the whole system will be the number of sections, say p, times the input reconstruction order n and thus the total delay will be pn. Each of the sections will then be an IIR filter with numerator order $2n+2$ and denominator order 2 and thus the total computational complexity (number of additions and multiplications) will be roughly $2np + 4p$.

Alternative way is to cascade the filters in continuous time by forming a $2p$ order state space model consisting of all the filters. The discretized model will then be an IIR filter with numerator order $2n + 2p$ and denominator order $2p$. This will lead to computational complexity of roughly $2n + 4p$, which is always less or equal to the computational complexity of cascading in discrete time. The total delay of the sections will also be only n.

Thus in final computational complexity point of view it would be preferable to cascade the filters in continuous time and discretize only once. However, the disadvantage of this approach is that the computation of the discrete filter coefficients is much harder with the $2p$ order state space model than with p models of order 2. And these coefficients need to be re-evaluated always when the parameters of the equalizer change, that is, when the knobs of the equalizer are turned. For this reason, here we have chosen to use p discretized second order filters instead of a single $2p$ order discretized filter.

C. Numerical Comparison of Equalizer Designs

Figure 5 shows the magnitude and phase responses of different designs for the following peaking filter type of equalizer, which was also used as an example in Orfanidis' article [9]:

- The sampling frequency is $f_s = 44100$ Hz.
- The peak frequency is $f_0 = 11025$ Hz, which in normalized scale corresponds to $\omega_0 = 0.5\pi$ rads/sample.
- The bandwidth in normalized scale is $\Delta\omega = 0.2\pi$ rads/sample, which corresponds to $Q = 2.5$.
- The peak gain is 12 dB (boost).

Figure 6 shows the responses of the filter designs, which approximate an equalizer with the same specifications as above, but the peak gain is -12 dB (cut). The following designs are shown in Figures 5 and 6:

- *Bilinear* is a conventional bilinear transformation based design (see, e.g., [9]).
- *Orfanidis* is the Nyquist matched design proposed in [9].
- *New (10)* is the design proposed in this article with input reconstruction order $n = 10$.
- *Analog* is the analog filter response.

The coefficients B_j of the new design were numerically evaluated using 10 steps of Simpson's integration rule. In the phase responses, before plotting, we have compensated the known n sample delay to get the different designs comparable.

As can be seen in the magnitude (upper) parts of Figures 5 and 6, the problem in the Bilinear design is that the

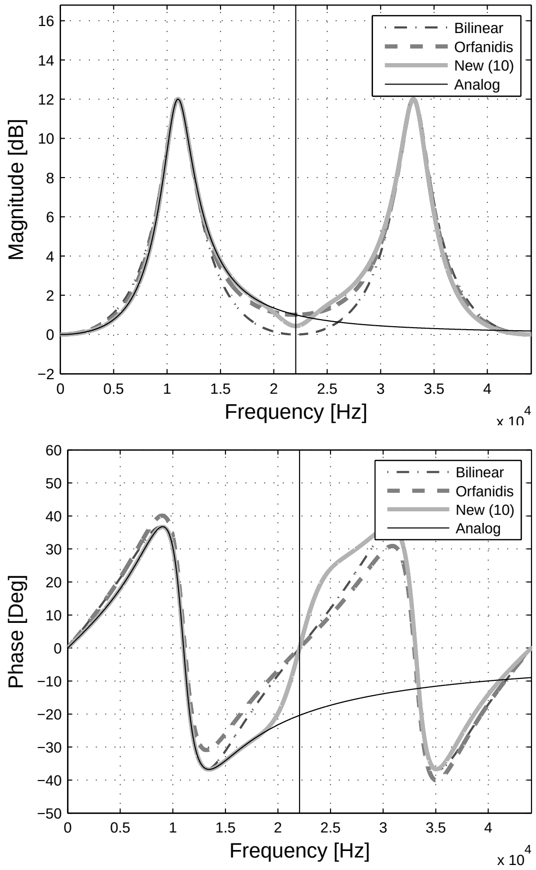


Fig. 5. Peaking equalizer designs with sampling frequency 44100 Hz, peak frequency 11025 Hz (0.5π rads/sample), quality factor $Q = 2.5$ (0.2π rads/sample) and peak gain 12 dB (boost).

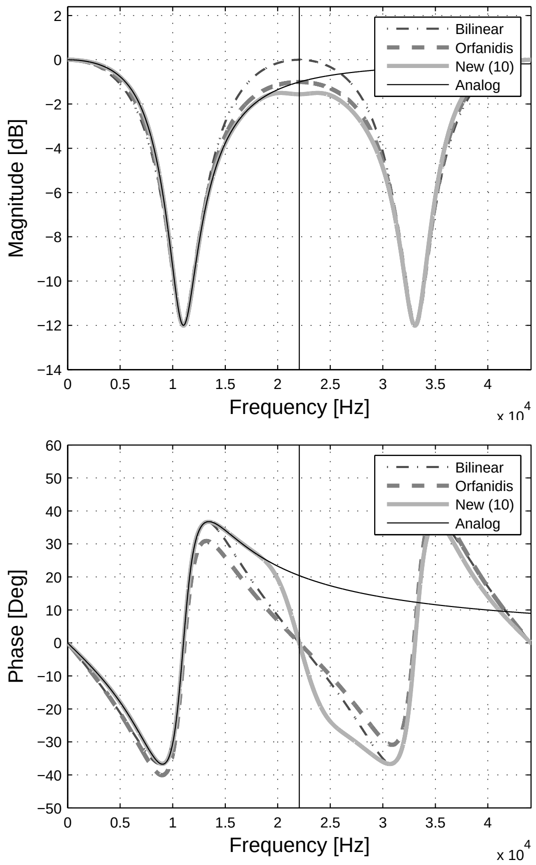


Fig. 6. Peaking equalizer designs with sampling frequency 44100 Hz, peak frequency 11025 Hz (0.5π rads/sample), quality factor $Q = 2.5$ (0.2π rads/sample) and peak gain -12 dB (cut).

magnitude response approaches zero too rapidly near the Nyquist frequency at 22500 Hz. The Orfanidis design is better in this sense, because the gain at Nyquist frequency has been matched to the analog gain and thus it stays closer to the analog design also near the Nyquist frequency. The New (10) design stays very close to the true response up to roughly 19 kHz and then curves towards zero.

The phase (lower) parts of the Figures 5 and 6 show that the problem in Bilinear and Orfanidis designs is that starting already roughly from 13 kHz the phase responses of the filters start significantly differ from the analog filter phase response. The New (10) filter performs better in this sense, because the phase response matches the analog response up to roughly 19 kHz and then curves towards zero.

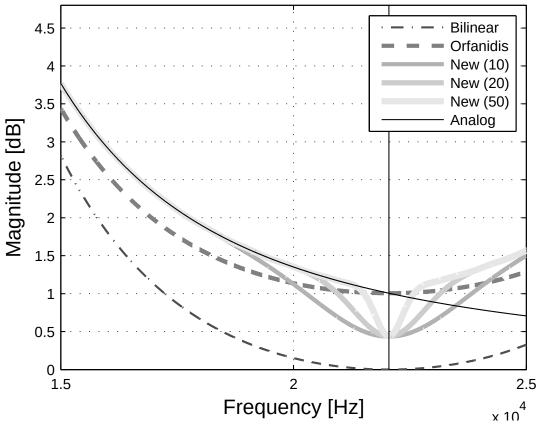
Figure 7 illustrates the effect of input reconstruction order in the proposed filter design. In the figure, New (10), New (20) and New (50) refer to the proposed designs with $n = 10$, $n = 20$ and $n = 50$, respectively. As can be seen in the figure, when the input reconstruction order increases, the magnitude response stays closer and closer to the analog design when the frequency approaches the Nyquist frequency. In the limit $n \to \infty$ we would get a filter, which would approximate the magnitude and phase responses perfectly up to Nyquist frequency and then dropped to zero exactly at the Nyquist frequency.

TABLE I
ROOT MEAN SQUARED ERRORS (RMSE) WITH RESPECT TO THE ANALOG FILTER IN MAGNITUDE AND PHASE RESPONSES OF DIFFERENT DESIGNS.

Design Method	Magn. 0-20kHz	Magn. 0-22.5kHz	Phase 0-20kHz	Phase 0-22.5kHz
Bilinear	0.1079	0.1112	5.0587	7.7662
Orfanidis	0.0384	0.0366	7.1368	9.2182
New (1)	0.2416	0.2317	7.0878	9.1820
New (5)	0.0210	0.0324	2.1909	5.8966
New (10)	0.0044	0.0210	0.4554	4.8430
New (20)	7.8844e-04	0.0152	0.0200	4.3921
New (50)	3.5433e-04	0.0101	0.0094	4.6826

The Table I shows the magnitude and phase errors between the discrete approximations and analog prototypes. The root mean squared errors (RMSE) have been computed at two different ranges: 0 Hz – 20000 Hz, which corresponds to the range that is normally considered to be the relevant band for audio applications. For completeness, we have also included errors computed from the frequency range from zero up to Nyquist frequency 22050 Hz.

As the table shows, the 1st order reconstruction has about twice the error of the bilinear transformation based method, but the 5th order input reconstruction is already on the same range of errors as Orfanidis' method. However, in the 5th order method, the phase error is much lower than that of Orfanidis' design. The 10th order input reconstruction is much better



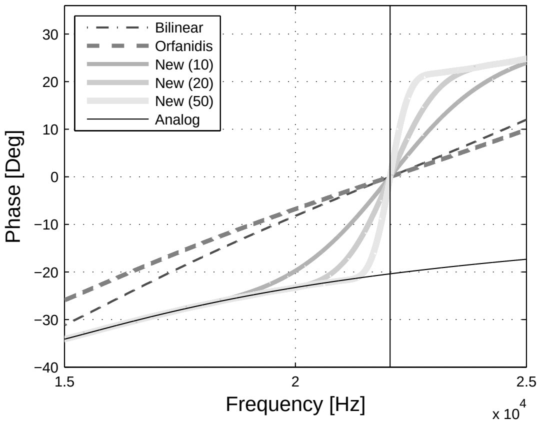


Fig. 7. Enlarged view of magnitude and phase responses of peaking equalizer designs, which illustrates the effect of input reconstruction order.

than any other method in both magnitude and phase responses. Increasing the reconstruction order lowers all the errors, expect that the phase response error, when measured from the whole range does not go down. This is because just below the Nyquist frequency there still is a small range where the phase differs from the analog prototype.

IV. Conclusion and Discussion

In this article, we have presented a general discretization method, which can be used for designing discrete versions of analog filters such that the frequency response of the digital design accurately matches that of the analog design. The approach is based on approximation of the input reconstruction with classical Shannon's interpolation theorem and numerical solving of the LTI differential equation with the reconstructed input. The resulting filter is a relatively low order IIR filter, which can be implemented in state space form or in direct IIR form. The proposed methodology has been applied to design of parametric equalizers and the frequency response of an example equalizer design has been compared to previously proposed equalizer design methods.

Although, the application example in this paper was the parametric equalizer design, the proposed methodology can be used in much wider range of applications. In audio signal processing, the proposed method can be used for converting any linear signal processing system, which has been designed in continuous time into a discrete-time system with approximately the same frequency and phase characteristics as the original system. These kind of conversions are needed, for example, in digital re-implementations of analog audio processing systems, where an important issue is not to destroy their unique processing characteristics. In telecommunication systems [17], filters are often formulated in continuous time and the proposed method provides the means to convert them into discrete filters while preserving the frequency and phase responses more accurately than the traditional methods. The recent rapid increase in computational power of processors has made digital implementation radio frequency (RF) filters feasible and thus the present methodology could be used for conversion of such analog systems into their digital counterparts.

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