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DESIGNING A NON-LINEAR TRACKING CONTROLLER FOR VEHICLE ACTIVE SUSPENSION SYSTEMS USING AN OPTIMIZATION PROCESS

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ABSTRACT−In this paper, a new non-linear tracking controller for vehicle active suspension systems is analytically designed using an optimization process. The proposed scheme employs a realistic non-linear quarter-car model, which is composed of a hardening spring and a quadratic damping force. The control input is the external active suspension force and is determined by minimizing a performance index defined as a weighted combination of conflicting objectives, namely ride quality, handling performance and control energy. A linear skyhook model with standard parameters is used as the reference model to be tracked by the controller. The robustness of the proposed controller in the presence of modeling uncertainties is investigated. The performed analysis and the simulation results indicate that both vehicle ride comfort and handling performance can be improved using the minimum external force when the proposed non-linear controller is engaged with the model. Meanwhile, a compromise between different objectives and control energy can easily be made by regulating their respective weighting factors, which are the free parameters of the control law.

KEY WORDS : Vehicle active suspension, Non-linear control, Optimization, Robustness analysis

1. INTRODUCTION

Vehicle suspension systems are mainly designed to isolate the car body and passengers from road irregularities. They also ensure good contact between the tires and road. Vibration isolation is required to ensure ride comfort, whereas road holding is important for the vehicle handling, which, in general, leads to enhanced safety. These are conflicting objectives, and a compromise between them should be taken into account when designing the vehicle suspension systems.

In a conventional passive suspension system, the fixed elasto-damping elements are selected to achieve a certain level of passenger comfort (Mirzaei and Hassannejad, 2007). However, there are limitations to providing both good ride comfort and suitable handling for widely changing road conditions.

To isolate the car body from road disturbances and to provide ride comfort, a soft suspension is required for the passive system. However, this leads to excessive wheel travel, which results in non-optimal tire attitude relative to the road and consequently causes poor handling.

The use of an active suspension system comprising a force-generating actuator between the sprung and un-sprung masses compensates for the drawbacks associated with passive systems. In the active suspension system, with proper control of the injected energy into the system, both ride quality and handling performance can be improved (Fateh and Alavi, 2009; Kim and Ro, 1998). The active suspension system may also be restricted due to practical constraints, such as control input energy. However, by minimizing the required energy and the implementation costs, these systems could be made more practical (Jalili and Esmaeizadeh, 2001). Therefore, the power limitation of actuators should be considered as an important factor in designing the active suspension system (Esmailzadeh and Taghirad, 1998).

The design problem for the active suspension involves finding a suitable external force control law as a product of an optimization process or a trade-off between conflicting objectives, namely ride comfort, road holding and suspension deflection. Additionally, the control law must minimize the required injected force. Thus, the weighted control force should be considered in the performance index as a minimized criterion, in addition to other criterion.

Linear optimal control strategies have been frequently used to design active suspension controllers (Esmailzadeh and Taghirad, 1998; Gopala Rao and Narayanan, 2008; Hrovat, 1993; Marzbanrad *et al*., 2004; Narayanan and Senthil, 1998; Zaremba *et al*., 1997). These methods can provide a systematic approach for achieving compromises

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between the conflicting objectives to fulfill the system requirements. However, these methods apply linear suspension models to determine the control law, whereas a real vehicle suspension has inherent nonlinearities. Kim and Ro compared the responses of linear and non-linear suspension models for different road inputs and concluded that the non-linear characteristics of suspension elements should be considered when developing a more accurate system model (Kim and Ro, 1998). Thus, they employed the sliding control method to design an active suspension controller because of its potential to address nonlinearities and intrinsic robustness.

There are other non-linear control laws for active suspension systems in the literature. A non-linear adaptive control approach was used by Alleyne and Hedrick (1995). Chein *et al*. (2009) presented feedback linearization control of nonlinear uncertain multi-input multi-output systems for tracking and almost disturbance decoupling performances. The other important non-linear methods used for active suspension systems are the following: feedback linearization and multi-layered feed-forward neural network (Li *et al*., 2010), impedance control (Fateh, 2009) and fuzzy sliding mode control (Leen *et al*., 2009).

In all the non-linear methods mentioned above, optimization is not used as the main procedure for determining the control laws. Gordon (1995) presented the optimal control problem for a nonlinear semi-active vehicle suspension system by applying Pontryagin's minimum principle and developed a numerical solution for it. Generally, the application of classic optimal control theories to non-linear systems requires the solution of the derived non-linear differential equations. It is very difficult or even impossible to find an analytical solution for these problems. In addition, numerical computation methods require online dynamic optimization and are difficult to solve and implement (Mirzaei *et al*., 2008; Mirzaeinejad and Mirzaei, 2010).

It should be noted that the designed controller for a vehicle suspension system should be robust with respect to modeling uncertainties, which are unavoidable due to variations of vehicle parameters or fading effects of un-modeled dynamics.

According to the requirements for the suspension system and considering the literature reviewed above, the necessity of designing a robust non-linear controller using the complete optimization problem with an analytical solution and the accompanying detailed investigations is apparent for the active suspension system. To achieve this aim, in the present study, a new optimal approach is applied to design a non-linear controller of suitable robustness for the vehicle active suspension system. This method, which directly employs a realistic non-linear quarter-car model, analytically leads to a closed-form control law has a suitable solution and can be easily implemented. In addition, online numerical computations in optimization are not required (Chen *et al*., 2003; Mirzaeinejad and Mirzaei, 2010; Slotine and Li, 1991). This method has been successfully developed for different automotive control systems by the authors (Mirzaei *et al*., 2008; Mirzaeinejad and Mirzaei, 2010). In the present application for an active suspension system, the capability of the designed controller in the presence of model nonlinearities and uncertainties is evaluated. Additionally, the optimal property of the control law in achieving the best compromise with respect to optimizing the conflicting objectives and control energy is investigated by the regulation of free parameters. Finally, the effectiveness of the proposed controller and its special cases in tracking the standard sky-hook model are compared with other conventional non-linear control methods.

2. MODELLING OF THE SUSPENSION SYSTEM

2.1. Design Model

In this section, a non-linear 2 DOF quarter-car model that captures the essential features of the vehicle suspension system is used to design the controller. This model employs realistic nonlinearities in the suspension spring and damper forces, which have important effects on the responses of the suspension system when excited by different road inputs. The design model and its free body diagram are shown in Figure 1. The equations of motion for the model are derived using Newton's second law as follows:

$$m_s\ddot{x}_s = -f_s - f_d + u \tag{1}$$

$$m_{us}\ddot{x}_{us} = f_s + f_d - f_{st} - f_{dt} - u \tag{2}$$

where x_s and x_{us} are the sprung and un-sprung mass displacements, f_s and f_d are the spring and damper forces and f_{st} and f_{dt} are the tire stiffness and damping forces, respectively. m_s is the sprung mass, and m_{us} is the un-sprung mass. Finally, u is the external force generated by the actuator, which must be determined from the control law.

In the linear suspension model, the spring and damper forces are described as linear functions of the suspension deflection and the suspension velocity, respectively. However, the linear region is relatively small in view of the ride characteristics. To achieve a model with higher

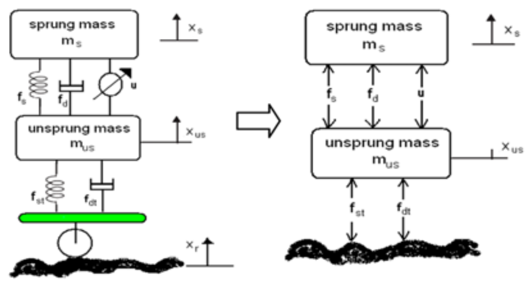

Figure 1. Quarter-car suspension system model and its free body diagram.

accuracy for all operating conditions, the spring force, f_s, and the damper force, f_d, are modeled as nonlinear polynomial functions, as presented by Kim and Ro (1998). They obtained these functions from the measured data for commonly used suspension elements as follows:

$$f_s = k_1\Delta x + k_2\Delta x^2 + k_3\Delta x^2 \tag{3}$$

$$f_d = c_1\Delta\dot{x} + c_2\Delta\dot{x}^2 \tag{4}$$

The tire elastic and damping forces are calculated as

$$f_{st} = k_{us}(x_{us} - x_r) \tag{5}$$

$$f_{dt} = c_{us}(\dot{x}_{us} - \dot{x}_r) \tag{6}$$

where k_{us} and c_{us} are the tire stiffness and damping coefficients, respectively. x_r is the road disturbance input.

The non-linear suspension system dynamics described by equations (1) and (2) can be written in the following standard state space form:

$$\dot{x}_1 = x_2 - x_4 \tag{7}$$

$$\dot{x}_2 = f_1 + u/m_s \tag{8}$$

$$\dot{x}_3 = x_4 - \dot{x}_r \tag{9}$$

$$\dot{x}_4 = f_2 - u/m_{us} \tag{10}$$

In the above equations, $x_1 = x_s$-x_{us}, $x_2 = \dot{x}_s$, $x_3 = x_{us}$-x_r and $x_4 = \dot{x}_{us}$ are the state variables of the system and indicate, respectively, the suspension deflection, the absolute sprung mass velocity, the tire deflection and the un-sprung mass absolute velocity. All elastic and damping forces have been incorporated in f_1 and f_2 as follows:

Table 1. Parameters for the case study suspension system.

Sprung mass, m_s	290 kg
Un-sprung mass, m_{us}	59 kg
Tire stiffness, k_{us}	190000 N/m
Tire damping, c_{us}	70 N.s/m
Spring force coefficient, k_1	12394 N/m
Spring force coefficient, k_2	-73696 N/m^2
Spring force coefficient, k_3	3170400 N/m^3
Damping force coefficient, c_1	1385 N/s
Damping force coefficient, c_2	524 N/s^2
Skyhook suspension stiffness, k_s	16812 N/m
Skyhook suspension damping, c_s	1000 N.s/m
Skyhook damping, c_{sh}	2500 N.s/m

$$f_1 = \left(-\frac{1}{m_s}\right)[f_s + f_d], \qquad f_2 = \left(\frac{1}{m_{us}}\right)[f_s + f_d - f_{st} - f_{dt}] \tag{11}$$

To emphasize the influence of the non-linear characteristics of the suspension model, the responses of the non-linear model and the corresponding linear model are compared in Figure 2. Note that the linear model is derived from linearizing the non-linear model about an operating point. The parameters of the case study suspension system are given in Table 1.

For the simulation, the road input, shown in Figure 2, is assumed to consist of two bumps: the first with a 10 cm height and a 0.25 s duration and the second, coming 0.75 s after the first, with a 7.5 cm height and a 0.25 s duration. The road profile as a function of time is given by

$$x_r = \begin{cases} 0.05(1-\cos 8\pi t) & 0.5 \le t \le 0.75 \\ 0.035(1-\cos 8\pi t) & 1.5 \le t \le 1.75 \\ 0 & other \end{cases} \tag{12}$$

As seen in Figure 2, the responses of the linear and non-

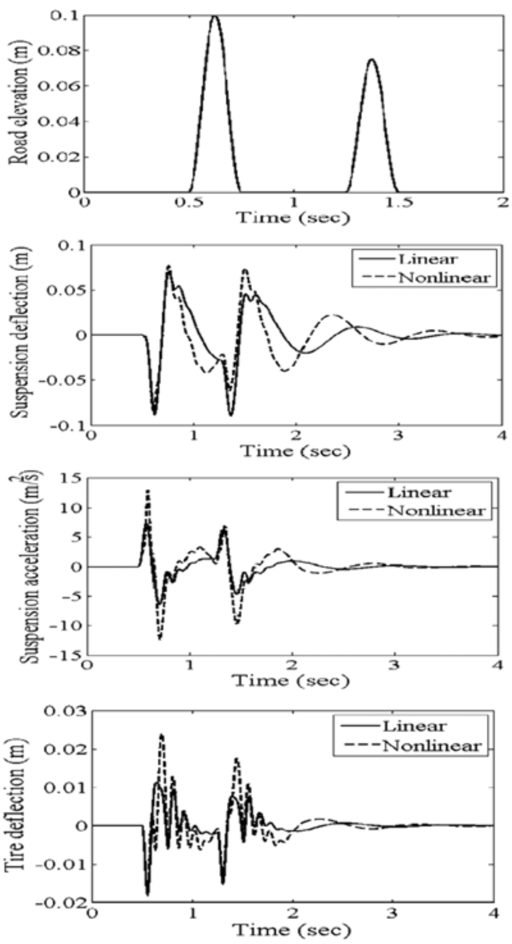

Figure 2. Comparison of the responses of the linear and non-linear models for the two-bump road input.

linear models are quite different from each other because the excitation amplitude is relatively large. The responses of the nonlinear model are large and need to be controlled. The same result with additional details has been presented by Kim and Ro (1998) for sinusoidal excitations with different amplitudes. They showed that when the road input is small, there is no considerable difference between the linear and nonlinear models. However, as the excitation amplitude increases, the responses of the two models become quite different from each other. This indicates that the force nonlinearities, which expand the valid range of operation, should be considered in developing a more accurate model from which a controller with enhanced performance can be designed.

2.2. Reference Model

As was considered in section 2.1, the nonlinearity of actual suspension element forces, especially when the excitation amplitude is large, deteriorates the system response when compared with the linear suspension model. The system uncertainties due to variations of the system parameters can also intensify this problem. Therefore, the well-known linear skyhook model, shown in Figure 3, with fix parameters is used as the reference model to be tracked by the designed controller (Kim and Ro, 1998; Hong *et al.*, 2002; Gopala Rao and Narayanan, 2009). This model, which consists of a passive damper connected between the sprung mass and a fictional fixed point in the inertial space, has more suitable responses than a conventional linear passive suspension. In fact, the simple state feedback control law, $u = c_{sh}\dot{x}_s$, for the linear passive suspension system, which reduces the sprung mass motion and consequently improves the ride quality, can be used to produce the desired skyhook damping effect.

In the same manner as the design model described in the previous section, the state space equations of the skyhook model are easily derived from Figure 3:

$$\dot{x}_{1d} = x_{2d} - x_{4d} \tag{13}$$

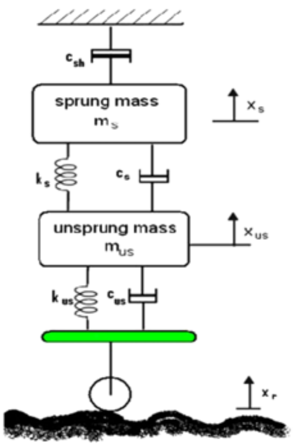

Figure 3. Skyhook model used as the reference model.

$$\dot{x}_2 = f_{1d} - c_{sh}x_{2d}/m_s \tag{14}$$

$$\dot{x}_{3d} = x_{4d} - \dot{x}_y \tag{15}$$

$$\dot{x}_4 = f_{2d} \tag{16}$$

where c_{sh} is the skyhook damping coefficient. All the state variables are defined similar to the design model. Additionally, the functions f_{1d} and f_{2d} have been defined as follows:

$$f_{1d} = \left(-\frac{1}{m_s}\right)[f_{sd} + f_{dd}] \tag{17}$$

$$f_{2d} = \left(\frac{1}{m_{us}}\right)[f_{sd} + f_{dd} - f_{std} - f_{dtd}] \tag{18}$$

Where the spring and damper forces, f_{ad} and f_{dd}, respectively, are considered to be the linear functions of the system states as follows:

$$f_{sd} = k_s x_{1d} \tag{19}$$

$$f_{dd} = c_s(x_{2d} - x_{4d}) \tag{20}$$

In the above equations, the subscript d indicates the desired model.

3. CONTROL SYSTEM DESIGN

The purpose of the control system is to keep the actual responses of the vehicle, such as sprung mass acceleration, $\dot{x}_2$, suspension deflection, x_1, and tire deflection, x_3, close to the desired responses of the skyhook model by using a minimum external control force, u. To achieve this aim, an optimization-based non-linear approach should be applied for calculating the external control force. Note that no suspension system can simultaneously minimize all mentioned tracking errors and the control energy. However, by employing an adjustable control law, a suitable trade-off between the conflicting objectives can be easily made.

3.1. Development of the Control Law

In this section, an optimal non-linear control law is developed for tracking the desired responses based on the prediction of the actual suspension system responses. According to the suspension system requirements described previously, three controlled state variables, x_1, x_2 and x_3, are considered as the outputs of the system:

$$y = [x_1\ x_2\ x_3]^T \tag{21}$$

It should be noted that for the state feedback control laws, the sprung mass absolute velocity, x_2, which is a state of the system, has been considered as a controlled variable that is a measure of ride comfort instead of the sprung mass acceleration (Kim and Ro, 1998; Esmailzadeh and Taghirad, 1998; Hong *et al*., 2002; Gopala Rao and Narayanan, 2008). Additionally, according to the prediction-based optimal control method used in this study, the relative degree of each predicted output has to be at least one to have a minimum control effort. However, the system degree relative to the sprung mass acceleration is zero. It will be shown in the simulation study that the sprung mass acceleration is perfectly controlled by controlling the absolute velocity.

Now, according to the proposed control method, the non-linear response of each output, $x_i(i=1,2,3)$, for the next time interval, $x_i(t+h)$, is first predicted by a Taylor series expansion, following which the current control force, $u(t)$, will be found based on continuous minimization of the predicted tracking errors. The predictive period, h, is a real positive number.

The performance index that penalizes the tracking errors in the next instance and the current control expenditure is considered in the following form:

$$J(u)=\frac{1}{2}\Sigma_{i=1}^{3}\rho_i e_i^2(t+h)+\frac{1}{2}\rho_4 u^2(t) \tag{22}$$

where $\rho_i>0$ and $\rho_4\geq 0$ are weighting factors indicating the relative importance of the corresponding terms. The optimal control problem is to find the current control force, $u(t)$, by minimizing the performance index (22) subject to the nonlinear state equations (7) ~ (10).

The next tracking errors, $e_i(t+h)$, are defined as follows:

$$e_i(t+h)=x_i(t+h)-x_{id}(t+h) \tag{23}$$

The following k_i th-order Taylor series expansion at t approximates each $x_i(t+h)$:

$$x_i(t+h)=x_i(t)+h\dot{x}_i(t)+\ldots+\frac{h^k}{k!}x_i^{(k)}(t) \tag{24}$$

The expansion order, k_i, for each output is chosen in a way that is suitable for the purpose of the controller design on the basis of predictions. To achieve a small control effort and to prevent complexity in derivation and implementation of the control law, the expansion order is limited to the relative degree of the non-linear system (Chen *et al*., 2003; Mirzaei *et al*., 2008; Mirzaeinejad and Mirzaei, 2010). This selection, which is related to the zero control order, eliminates the derivatives of the control input in the prediction of each output and results in adequate performance for the non-linear system with a lower relative degree (Chen *et al*., 2003; Mirzaei *et al*., 2008; Mirzaeinejad and Mirzaei, 2010). Thus, the control effort will be constant in the prediction interval

$$\frac{d}{d\tau}u(t+\tau)=0 \quad for\ \tau\in[0,h] \tag{25}$$

The relative degree is determined as the lowest order of the derivative of each output, $x_i(i=1,2,3)$, in which the input, $u(t)$, first appears explicitly. According to the plant state equations (7) ~ (10), the relative degree of the system with respect to x_1 and x_3 is r = 2 and with respect to x_2 is r = 1. Thus, a first-order Taylor series expansion is sufficient for x_2, but a second-order (or higher-order) Taylor series expansion is required for x_1 and x_3. It follows that the non-linear response of each output for the next time interval is predicted as follows:

$$x_1(t+h)=x_1+h\dot{x}_1+\frac{h^2}{2}\left(f_1-f_2+\left(\frac{1}{m_s}+\frac{1}{m_{us}}\right)u\right) \tag{26}$$

$$x_2(t+h)=x_2+h\left(f_1+\frac{u}{m_s}\right) \tag{27}$$

$$x_3(t+h)=x_3+h\dot{x}_3+\frac{h^2}{2}\left(f_2-\frac{u}{m_{us}}-\ddot{x}_r\right) \tag{28}$$

In the above equations, the state equations (7) to (10) have been incorporated.

The desired outputs, $x_{id}(t+h)$ (i=1,2,3), can also be expanded in the same manner as the system outputs. By substituting the expanded desired outputs together with the expanded system outputs into (23), the next tracking errors are obtained as a function of the current control force. Therefore, the expanded performance index (22) can be obtained as a function of the control input by substituting the predicted error terms. The necessary condition for optimality is

$$\frac{\partial J}{\partial u}=0 \tag{29}$$

which leads to

$$\Sigma_{i=1}^{3}\rho_i e_i(t+h)\frac{\partial e_i(t+h)}{\partial u}+\rho_4 u(t)=0 \tag{30}$$

Solving the above equation for u yields

$$u=a\{\rho_1 b_1[e_1+h(e_2-e_4)+0.5h^2(f_1-f_2-\ddot{x}_{1d})]+$$
$$(31)\rho_2 b_2[e_2+h(f_1-\dot{x}_{2d})]+\rho_3 b_3[e_3+he_4+0.5h^2(f_2-\ddot{x}_{ad})]\}$$

where

$$a=1/\{\rho_1 b_1^2+\rho_2 b_2^2+\rho_3 b_3^2+\rho_4\} \tag{32}$$

$$b_1=\frac{h^2}{2}\left(\frac{1}{m_s}+\frac{1}{m_{us}}\right) \tag{33}$$

$$b_2=h/m_s \tag{34}$$

$$b_3=\frac{k^2}{2m_{us}} \tag{35}$$

3.2. Evaluation of the Control Law

In this section, the main properties of the control law (31) are investigated, and its advantages over other conventional control approaches are discussed. The derived control law is given in analytical closed form, which is mathematically tractable. It is easy to solve and implement, and online numerical computations in optimizations are not required. Optimality of the control law provides the possibility of minimizing the external control force, at the cost of some admissible tracking errors, by increasing the weighting factor of the control input, ρ_4. Meanwhile, a suitable trade-off between the conflicting objectives can be provided by regulation of the weighting factors ρ_1, ρ_2 and ρ_3 as the free parameters. The greater the weighting factor, the better the tracking for the corresponding output. The exact tracking for each output can be achieved by setting the other weighting factors to zero. For example, when the ride comfort is considered as the only design criterion, the special case of the control law (31), which tracks only the sprung mass motion with limited control force, is derived by applying $\rho_1 = \rho_3 = 0$ as follows:

$$u = -\frac{\kappa_2}{h}[e_2 + h\hat{f}_1 - h\dot{x}_{2d}] \quad (36)$$

where

$$\kappa_2 = \frac{\rho_2 b_2^2}{\rho_2 b_2^2 + \rho_4} \quad (37)$$

is defined as the reduction factor due to the limitation of the control force. According to equation (37), if $\rho_4 = 0$, then $\kappa_2 = 1$. This case is considered as the cheap control in which there is no limitation on the control input. By increasing the weighting factor of the control input, ρ_4, the external control force, u, is decreased at the cost of some tracking errors (Mirzaeinejad and Mirzaei, 2010). To further illustrate the main properties of this special version of the designed controller, the control law (36) can be substituted into the actual model (8)

$$\dot{x}_2 = f_1 - \frac{\kappa_2}{h}[e_2 + h\hat{f}_1 - h\dot{x}_{2d}] \quad (38)$$

Note that the symbol "^" denotes the nominal model. Deviation of f_1 from the nominal model, $\hat{f}_1$, is related to the modeling uncertainties. The tracking error of the sprung mass motion is then derived by manipulating equation (38) as follows:

$$\dot{e}_2 + \frac{\kappa_2}{h}e_2 = (f_1 - \hat{f}_1) + (1 - \kappa_2)(\hat{f}_1 - \dot{x}_{2d}) \quad (39)$$

where $e_2 = x_2 - x_{2d}$ is the current tracking error of the sprung mass velocity. The right hand side of Equation (39) arises from both modeling uncertainty and reducing the control input. These terms will always lead to some tracking errors. When there is no reduction in the control input or modeling uncertainty (i.e., $\rho_4 = 0$ and $f_1 - \hat{f}_1 = 0$), the tracking error dynamics of the sprung mass velocity can be obtained as follows:

$$\dot{e}_2 + \frac{1}{h}e_2 = 0 \quad (40)$$

It is clear that the closed-loop system is linear and exponentially stable for any $h > 0$. This version of the derived control law naturally leads to a special case of input/output linearization. According to the error dynamics (40), the perfect tracking is maintained for all $t \in [0, t_f]$ because the initial error is zero.

The tracking error that arises due to modeling uncertainty can be controlled by regulation of the prediction time, h, as another free parameter (Mirzaei *et al*., 2008; Mirzaeinejad and Mirzaei, 2010). A high degree of robustness in the presence of some modeling uncertainties can be achieved for small values of h.

In the general case, according to the active suspension requirements, considering the weighting factors ρ_1 and ρ_3 in calculating the external force is required to limit the suspension deflection and the tire deflection, respectively. However, regarding ρ_4, applying these factors causes the sprung mass motion to increase and, consequently, the ride comfort to decrease. How to adjust the free parameter of the controller will be further investigated in simulation studies conducted in the next section.

4. SIMULATION RESULTS

Simulation studies are conducted to show the effectiveness of the proposed control system. In addition, the influence

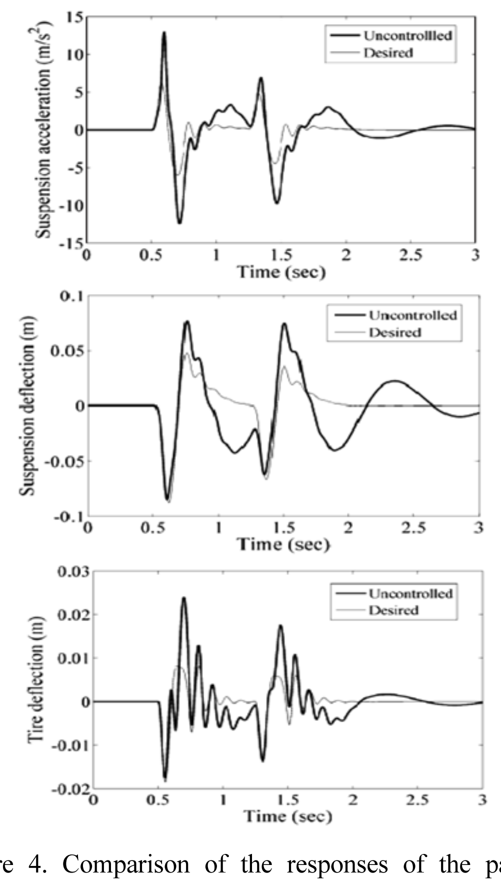

Figure 4. Comparison of the responses of the passive (uncontrolled) system and the desired skyhook model.

of the free parameters of the control law on the system responses and control energy are investigated in this section. The parameters of the suspension model and the road input specifications considered for the simulation study were described in section 2.

To emphasize the necessity of controlling the suspension system, the responses of the uncontrolled system with nonlinearities and the desired skyhook model are compared in Figure 4. The results indicate a considerable difference between the responses of the two models, which needs to be minimized. Both ride quality and handling of the actual passive suspension system should be improved.

The responses of the actual suspension system and the desired model strongly depend on the amplitude of the road excitations. In this study, to show the effectiveness of the designed control system, the road input is assumed to have two bumps with relatively large amplitudes (a severe condition). Obviously, when a bump with a smaller height is chosen, the sprung mass acceleration of the desired model would be less than that predicted by the present result.

The primary purpose of controlling the suspension system is to ensure ride comfort. Limiting tire deflection to improve road holding and limiting suspension deflection are of secondary and tertiary importance. The weighting factor of each characteristic considered in the proposed control law determines its importance relative to the others. To investigate the influence of the weighting factors on the system responses, Figure 5 shows the simulation results for the case when only the ride quality is taken into account in the performance index, i.e., $\rho_2 = 1$, $\rho_1 = \rho_3 = \rho_4 = 0$. As explained through equation (40), the acceleration of the sprung mass is exactly tracked by the controller as a result of controlling the absolute velocity of the sprung mass. However, there are tracking errors associated with the suspension deflection and tire deflection. In particular, the tire deflection response does not converge to the steady state in finite time, and it remains severely oscillatory. In other words, the road stability of the vehicle is unsafe. This causes the control input to be oscillatory as well.

To provide good ride comfort and suitable road stability with a smooth control force, the suspension travel and tire deflection should be accounted for in the performance index, along with the sprung mass motion as the controlled parameters. The amount of each weighting factor in the control law is determined according to its relative importance through a trial and error procedure. Due to the great importance of ride quality, ρ_2 is first selected as a suitably large value. Then, the other weighting factor is gradually increased to the extent that a suitable response, without any undesired oscillations, is achieved. Figure 6 shows the best possible responses, found after a trial and error procedure. These responses represent the best compromise between the comfort and stability. As seen in Figure 6, the undesired oscillations in the previous responses have been removed, and all responses are satisfactory.

Another important advantage of the proposed controller is the possibility of limiting the external control force to an acceptable range by the regulation of ρ_4. In all the cases discussed above, it is assumed that there is no limitation on the control energy (i.e., the cheap control case, $\rho_4 = 0$). As is seen in Figure 6, the peak of the control force reaches 4000 N, which may not be generated in practice. For the active suspension system, the maximum force generated by the actuator is reported as 3500 N (Esmailzadeh and Taghirad, 1998).

To limit the control energy consumed, the control

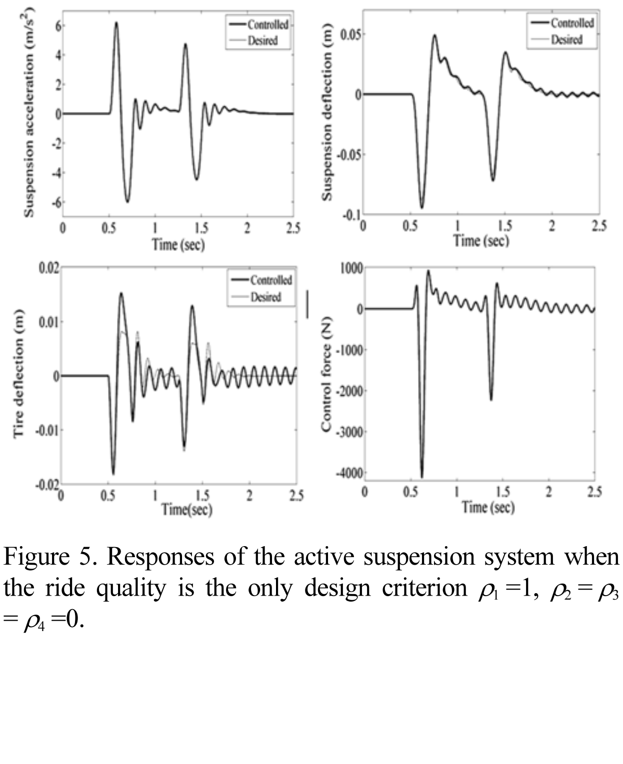

Figure 5. Responses of the active suspension system when the ride quality is the only design criterion $\rho_1 = 1$, $\rho_2 = \rho_3 = \rho_4 = 0$.

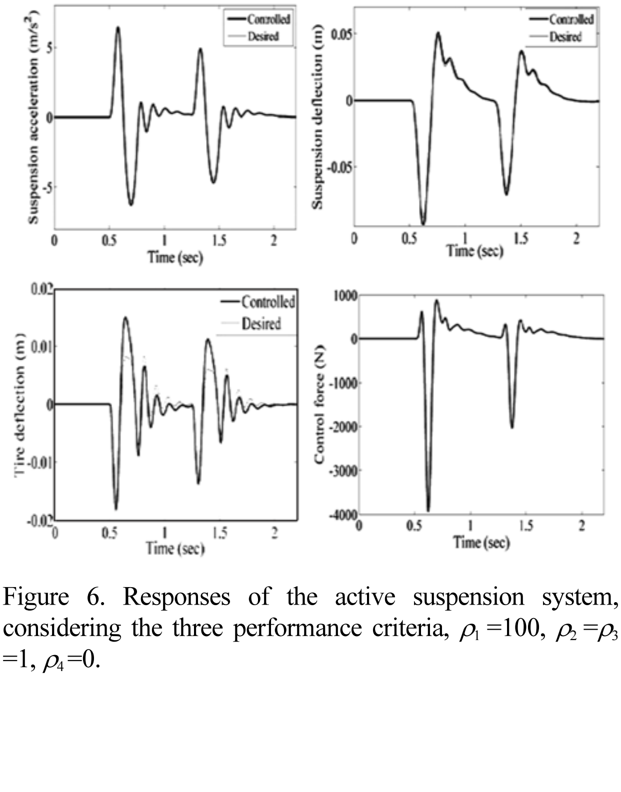

Figure 6. Responses of the active suspension system, considering the three performance criteria, $\rho_1 = 100$, $\rho_2 = \rho_3 = 1$, $\rho_4 = 0$.

weighting factor, ρ_4, should be increased (i.e., the expensive control case). Figure 7 compares the system responses and calculated control forces for different values of the weighting factor, ρ_4. The obtained results indicate that by increasing ρ_4, the control force can be reduced and maintained below the an admissible value at for some acceptable level of tracking error.

As considered in Figure 7, the peak of the control force through $\rho_4 = 2e{-}9$ has become nearly half of that for $\rho_4 = 0$. However, the tracking errors caused by this control weighting factor are still acceptable. It should be noted that to decrease the control input, the value of the weighting factor, ρ_4, can be increased, to some extent. Otherwise, the system responses cannot follow the behavior of the reference model adequately, and the system remains uncomfortable and unsafe.

Finally, the performance of the proposed controller is compared with that of a sliding mode controller reported in the literature (Kim and Ro, 1998). Both controllers use the same non-linear suspension model as the design model, but the sliding controller tracks only the sprung mass motions to improve the ride comfort. Figure 8 compares the responses of two controllers, the present controller and the previous sliding controller.

As seen in from Figure 8, although both controllers perfectly track the suspension acceleration, the tire deflection response of the sliding controller does not converge to the steady state at a finite time, as in Figure 5. This is due to the fact that the limitation associated with tire deflection has not been considered in the design of the sliding controller. The control input is oscillatory as well. Note that because of the robust non-linear characteristic of the sliding controller, the special case of the optimal non-linear controller proposed in this study is adopted for the comparison. However, unlike the sliding controller, the proposed optimal nonlinear controller provides the possibility of reducing the control force in an acceptable range, as shown in Figure 7.

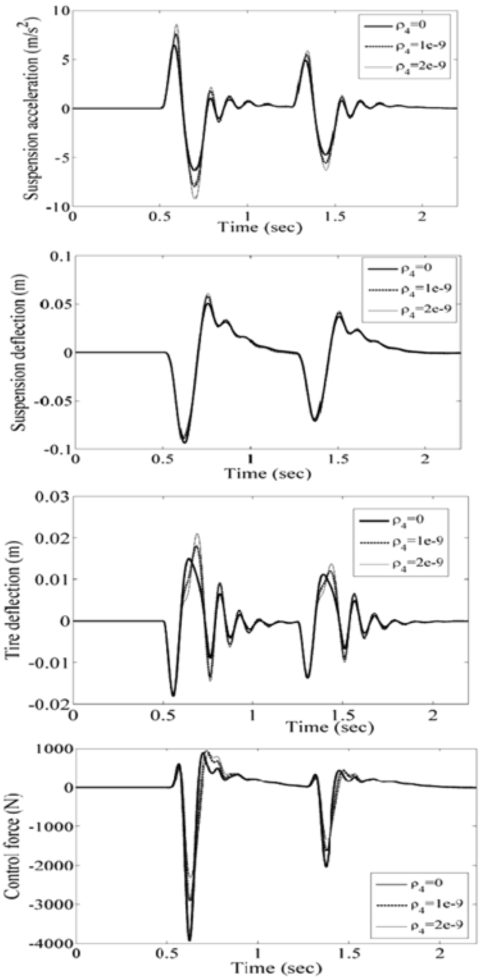

Figure 7. Effect of the control weighting factor on the active suspension system responses.

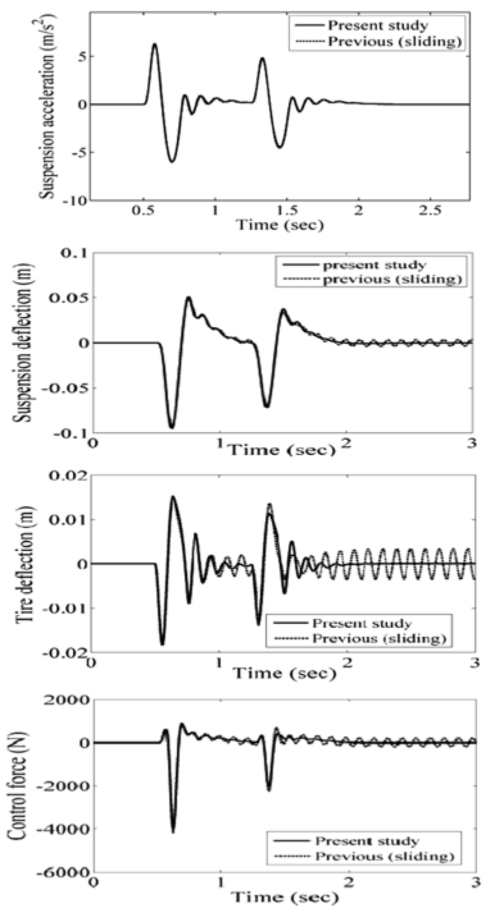

Figure 8. Comparison of the performances of the proposed controller and the sliding mode controller.

5. CONCLUSION

In this paper, an optimization-based non-linear control approach is developed to control a quarter-car suspension system model with realistic nonlinearities in the spring and damper forces. By considering different objectives in the performance index, both ride quality and road stability are improved with minimal external control force in the presence of modeling uncertainties. Additionally, by adjusting the weighting factor of the control input, the physical limitation of the actuator can be satisfied smoothly. The proposed optimal nonlinear control law is given in an analytical form, which can be easily extended to the comprehensive vehicle dynamics models as in those in (Huang *et al*., 2009).

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