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Application of Chebyshev polynomials to derive efficient algorithms for the solution of optimal control problems

B. Kafash [a,*], **A. Delavarkhalafi** [a], **S.M. Karbassi** [b]

[a] Faculty of Mathematics, Yazd University, Yazd, P.O. Box 89197/741, Iran
[b] Faculty of Advanced Education, Islamic Azad University of Yazd, Yazd Branch, Yazd, P.O. Box 89195/155, Iran

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Abstract In this paper, new and efficient algorithms for solving optimal control problems and the controlled Duffing oscillator are presented. The solution is based on state parameterization, such that the state variable can be considered as a linear combination of Chebyshev polynomials with unknown coefficients. First, an optimization problem in $(n+1)$-dimensional space is changed into a one-dimensional optimization problem, which can then be solved easily. By these algorithms, the control and state variables can be approximated as a function of time. Convergence of the algorithms is proved and some illustrative examples are presented to show the efficiency and reliability of the presented method.

1. Introduction

Optimal control is a mathematically challenging and practically significant discipline. It has many successful practical applications in a wide range of disciplines, such as engineering, economics and finance, to name just a few. In recent years, considerable attention has been given to the use of spectral methods for the solution of nonlinear physical problems. The controlled Duffing oscillator describes many such oscillatory phenomena in nonlinear engineering systems. The controlled Duffing oscillator, in particular, has received a considerable amount of attention in recent decades.

In practice, many optimal control problems are subject to constraints in state and/or control variables. In direct methods, the optimal solution is obtained by direct minimization of the performance index, subject to constraints. Dynamic optimization programming and Pontryagin's maximum principle method [1–5] represent the best known methods for solving optimal control problems. In [6], the computation of switching surfaces in general time optimal control problems, using a polynomial equations system, was considered. In [7], a general framework is constructed, upon which an explicit parametric formula can be derived for state feedback controllers, containing all possible combinations of parameter. As analytical solutions for problems of optimal control are not always available, finding an approximate solution is at least the most logical way to solve them. The study of numerical methods has provided an attractive field for researchers of mathematical sciences, which has given rise to the appearance of different numerical computational methods and efficient algorithms in solving optimal control problems (for details see [8–14]). In particular, the control parametrization technique is used in [9,15], while the Control Parametrization Enhancing Technique (CPET) is introduced in [15,16]. A class of constrained optimal control problems subject to canonical constraints was considered in [17]. Vlassenbroec presented a numerical technique for solving non-linear constrained optimal control problems [18]. Jaddu presented numerical methods to solve unconstrained and constrained optimal control problems [19] and later, extended his ideas to solve nonlinear optimal control problems with terminal state constraints, control inequality constraints and simple bounds on state variables [20]. Lee et al. have solved the optimal control and optimal parameter selection problems of a rotating flexible beam fully covered with Active Constrained Layer Damping (ACLD) treatment with a computational approach [21]. Van Dooren and Vlassenbroech [22] introduced a direct method for the controlled Duffing oscillator. El-Gindy [23] presented

* Corresponding author.
E-mail address: bkafash@stu.yazduni.ac.ir (B. Kafash).
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an alternative technique for solving controlled Duffing oscillator problems, which is based on the El-Gendi method [24]. This method starts with a Chebyshev approximation for the highest order derivative, and generates approximations to the lower order derivatives through successive integrations. In [25], a numerical technique is presented for solving the controlled Duffing oscillator in which the control and state variables are approximated by the Chebyshev series.

State parametrization converts the problem to a non-linear optimization problem and finds $(n+1)$ unknown polynomial coefficients of degree, at most, n, in the form of $\sum_{k=0}^{n} a_k t^k$ for an optimal solution [13,26]. One may use, for example, geometric or exponential combinations. There is an optimal control software package, MISER3, which has been developed by Jennings et al. [27,28] to solve optimal control problems. MISER3 has been used in solving various kinds of control problem with different aspects [29].

In this paper, the algorithm presented in [13] is modified, and an efficient iterative algorithm is obtained. In this way, only one unknown coefficient is calculated for finding a suitable approximation; further iterations leads to favorable accuracy. The same computational technique can be extended to solve the controlled Duffing oscillator. In addition, by the proposed algorithm, the control and state variables can be approximated as a function of time.

2. Chebyshev polynomials

In this section, Chebyshev polynomials, which are used in the next sections, are reviewed briefly [30,31].

Definition 1. The Chebyshev polynomial, $T_n(t)$, of the first kind is a polynomial in t of degree n defined by the relationship:

$$T_n(t) = \cos(n \cos^{-1} t), \tag{1}$$

where:

$$t = \cos\theta. \tag{2}$$

If the range of variable t is the interval, $[-1, 1]$, then the range of the corresponding variable, θ, can be taken as $[0, \pi]$. These ranges are traversed in opposite directions, since $t = -1$ corresponds to $\theta = \pi$, and $t = 1$ corresponds to $\theta = 0$. It is well known (as a consequence of de Moivre's Theorem) that $\cos n\theta$ is a polynomial of degree n in $\cos\theta$, and, indeed, one is familiar with the elementary formulae:

$$\begin{aligned}
&\cos 0\theta = 1, \qquad \cos 1\theta = \cos\theta,\\
&\cos 2\theta = 2\cos^2\theta - 1,\\
&\cos 3\theta = 4\cos^3\theta - 3\cos\theta,\\
&\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1, \ldots.
\end{aligned} \tag{3}$$

It can be immediately deduced from Eq. (2) that the first few Chebyshev polynomials are:

$$\begin{aligned}
&T_0(t) = 1, \qquad T_1(t) = t, \qquad T_2(t) = 2t^2 - 1,\\
&T_3(t) = 4t^3 - 3t, \qquad T_4(t) = 8t^4 - 8t^2 + 1, \ldots.
\end{aligned} \tag{4}$$

In practice, it is neither convenient nor efficient to work out each $T_n(t)$ from the first principles. Rather, by combining the trigonometric identity:

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta, \tag{5}$$

with Eqs. (1), (2) and (4), clearly the fundamental recurrence relationship can be obtained as:

$$\begin{cases} T_0(t) = 1, & T_1(t) = t,\\ T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t), & n = 1, 2, 3, \ldots, \end{cases} \tag{6}$$

which generates all Chebyshev polynomials, $\{T_n(t)\}$, efficiently.

Lemma 1. *In Chebyshev polynomials, all even powers of t are even functions, and all odd powers of t are odd functions.*

Proof. Eqs. (4) suggest that $T_n(t)$ is an even or odd function involving only even or odd powers of t, according to n being even or odd. This may be deduced rigorously from Relations 6 by induction. □

Lemma 2. *It can be inferred that:*

$$T_n(1)T_{n+1}(-1) - T_n(-1)T_{n+1}(1) \neq 0. \tag{7}$$

This is later used in Algorithm 2.

Proof. For the boundary points, we have $\theta = \pi$ and $\theta = 0$, so that corresponding values of $T_n(t)$ in these points are $(-1)^n$ and 1, respectively. From the above explanations and Lemma 1:

If n is even, then:

$$\begin{aligned}
&T_n(1)T_{n+1}(-1) - T_n(-1)T_{n+1}(1) = -T_n(1)T_{n+1}(1)\\
&-T_n(1)T_{n+1}(1) = -2 \neq 0,
\end{aligned}$$

and if n is odd, then:

$$\begin{aligned}
&T_n(1)T_{n+1}(-1) - T_n(-1)T_{n+1}(1) = T_n(1)T_{n+1}(1)\\
&+T_n(1)T_{n+1}(1) = 2 \neq 0. \quad \square
\end{aligned}$$

Function $f(t)$ can be approximated by a Chebyshev series of length N as follows [19]:

$$f(t) = \frac{a_0}{2} + \sum_{i=1}^{N} a_i T_i(t),$$

where:

$$a_j = \frac{2}{K}\sum_{i=1}^{N} f(\cos(\theta_j))\cos(j\theta_i), \quad j = 0, 1, \ldots, N,$$

where:

$$\theta_j = \frac{2i-1}{2K}\pi, \quad i = 1, 2, \ldots, K,$$

and $K > N$.

The derivative of $f(t)$ with respect to t is given by:

$$\dot{f}(t) = \frac{b_0}{2} + \sum_{i=1}^{N-1} b_i T_i(t),$$

where:

$$\begin{aligned}
&b_{N-1} = 2Na_N,\\
&b_{N-2} = 2(N-1)a_{N-1},\\
&b_{r-1} = b_{r+1} + 2ra_r, \quad r = 1, 2, \ldots, N-2.
\end{aligned}$$

Chebyshev polynomials can also be expressed in powers of t, and vice versa. For example, the powers of t can be expressed in terms of the Chebyshev polynomials of degrees up to n as below [31]:

$$t^n = 2^{1-n}\sum_{k=0}^{[n/2]} \binom{n}{k} T_{n-2k}(t), \tag{8}$$

where:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \tag{9}$$

and the dash ($\sum'$) denotes that the kth term in the summation is to be halved if n is even and $k = \frac{n}{2}$.

For $T_n(t)$, in terms of $t^n, t^{n-2}, t^{n-4}, \ldots$, the relevant result is obtained in the form [32]:

$$T_n(t) = \sum_{k=0}^{[n/2]} \left[(-1)^k \sum_{j=k}^{[n/2]} \binom{n}{2j} \binom{j}{k} \right] t^{n-2k}. \tag{10}$$

However, a simpler formula is given, for example, by Clenshaw (1962) and Snyder (1966), in the form [31]:

$$T_n(t) = \sum_{k=0}^{[n/2]} c_k^{(n)} t^{n-2k}, \tag{11}$$

where:

$$c_k^{(n)} = (-1)^k 2^{n-2k-1} \left[2\binom{n-k}{k} - \binom{n-k-1}{k} \right],$$

$$(2k < n),$$

and:

$$c_k^{(2k)} = (-1)^k, \quad (k \geq 0).$$

Also, the exact relation between the Chebyshev function and its derivatives is expressed as in [25]:

$$T_n(x) = \sum_{m=0}^{k} \frac{(-1)^m \binom{k}{m}}{2^k \chi_m} T_{n+k-2m}^{(k)}(x), \quad n > k, \tag{12}$$

where:

$$\chi_m = \prod_{\substack{j=0 \\ j \neq k=m}}^{k} (n+k-m-j).$$

Corollary 1. *Let Q_n' be the class of combinations of Chebyshev polynomials of degrees up to n, then:*

$$Q_{n+1}^{/} \supseteq Q_n'.$$

Proof. If Q_n' is the class of Chebyshev polynomials in t of degree n, then the result follows from Eqs. (8) to (11) immediately. □

3. Mathematical formulation

Consider the process described by the following system of nonlinear differential equations on the fixed time interval $[t_0, t_1]$:

$$U(\tau) = f(\tau, X(\tau), \dot{X}(\tau)), \tag{13}$$

with initial conditions:

$$X(t_0) = x^0, \qquad X(t_1) = x^1, \tag{14}$$

where $X(\cdot) : [t_0, t_1] \to \mathbb{R}$ is the state variable, $U(\cdot) : [t_0, t_1] \to \mathbb{R}$ is the control variable, and f is a real-valued continuously differentiable function. The problem of optimal control is then to find control $U(\cdot)$, transferring System 13 from position $X(t_0) = x^0$ to position $X(t_1) = x^1$ within the time $(t_1 - t_0)$, and yielding the optimum performance index, J, which is given by:

$$J = \int_{t_0}^{t_1} L(\tau, X(\tau), U(\tau)) d\tau. \tag{15}$$

We always assume that there are admissible controls that pass through (t_0, x^0) and (t_1, x^1). In this set of controls, we search for the control variable which minimizes J and call it optimal control. If $t_0 \neq -1$ or $t_1 \neq 1$, then for using Chebyshev polynomials, we introduce the transformation:

$$\tau = \frac{t_1 - t_0}{2} t + \frac{t_1 + t_0}{2}. \tag{16}$$

This change in variable produces $t \in [-1, 1]$, corresponding to $\tau \in [t_0, t_1]$. Using Eq. (16), the optimal control problem in Eqs. (13)–(15) can be obtained as follows.

The optimal control:

$$u(t) = f\left(\frac{t_1 - t_0}{2} t + \frac{t_1 + t_0}{2}, x(t), \dot{x}(t) \right), \tag{17}$$

and its corresponding trajectory $x(t)$ with initial conditions:

$$x(-1) = x^0, \qquad x(1) = x^1, \tag{18}$$

minimizes:

$$J(x) = \frac{t_1 - t_0}{2} \int_{-1}^{1} L\left(\frac{t_1 - t_0}{2} t + \frac{t_1 + t_0}{2}, x(t), u(t) \right) dt. \tag{19}$$

4. State parametrization

Let $Q \subset C^1([t_0, t_1])$ consisting of all functions passing through (t_0, x^0) and (t_1, x^1). From Eqs. (13) and (15), the performance index may be considered as a function of $X(\cdot)$; $J(X(\cdot))$. Then, the above optimal control problem may be interpreted as a minimization of J on set Q.

Let Q_n be a subset of Q, consisting of all polynomials of degree, at most, n, as:

$$X_n(\tau) = \sum_{k=0}^{n} a_k \tau^k, \tag{20}$$

where $n = 1, 2, \ldots$.

Now, consider the minimization of J on Q_n, with $\{a_k\}_{k=0}^n$ as unknowns. This is an optimization problem in $(n+1)$-dimensional space and $J(X_n)$ may be considered as $J(a_0, a_1, \ldots, a_n)$.

Theorem 1 (*Weierstrass Approximation Theorem (1885)*). *Let $f \in C([a, b], \mathbb{R})$. Then, there is a sequence of polynomials, $P_n(x)$, that converges uniformly to $f(x)$ on $[a, b]$.*

Proof. See [33]. □

Theorem 2. *If $\alpha_n = \inf_{Q_n} J$, for $n = 1, 2, \ldots$, then $\lim_{n\to\infty} \alpha_n = \alpha$, where $\alpha = \inf_Q J$.*

Proof. Let $\hat{\alpha} > J(X(\cdot))$ be the limit of non-increasing sequence $\{\alpha_n\}$. If $\hat{\alpha} > \alpha$, then $\varepsilon = \frac{\hat{\alpha} - \alpha}{2} > 0$, so;

$$\exists X(\cdot) \in Q, \quad J(X(\cdot)) < \varepsilon + \alpha = \frac{\hat{\alpha} - \alpha}{2} + \alpha = \frac{\hat{\alpha} + \alpha}{2} < \hat{\alpha},$$

such that $\hat{\alpha} > J(X(\cdot))$, which contradicts the continuity of J and Theorem 1. □

The above result is summarized in the following algorithm.

Algorithm 1. Here, the algorithm in [13] is generalized in order to consider $\tau \in [t_0, t_1]$.

Object: To obtain an optimal value for $J(\cdot)$.

Step 1. Choose an $\varepsilon > 0$.

Step 2. Let $n = 1, X_1(\tau) = x^0 + (x^1 - x^0) \frac{\tau - t_0}{t_1 - t_0}$ and $\alpha_1 = J(X_1(\cdot))$.

Step 3. Let $n \to n+1$ and find $\alpha_n = \inf_{Q_n} J$.
Step 4. If $|\alpha_{n-1} - \alpha_n| < \varepsilon$ then stop, otherwise return to Step 3.

In the above algorithm, the solution of an optimization problem in all iterations is required, and the solution of the iteration is not used to construct the next one.

This seems too expensive from a computational viewpoint. In the next section, this algorithm is improved.

5. Modified algorithm

In this section, we use a state parametrization method to derive a robust method for solving optimal control problems, numerically. In comparison with other numerical methods, the number of unknowns of the proposed method is lower than that in control state parametrization. In state parametrization, the solution of the approximation is considered as Eq. (20), using $(n+1)$ terms of $1, \tau, \tau^2, \ldots, \tau^n$ as a basis for Q_n. This is a poor choice for numerical calculations, but Chebyshev polynomials operate as a good basis. Algorithm 1 yields a solution of the optimization problem in all iterations, but the solution of each step is obtained independently from previous steps, so it is costly. Algorithm 1 is made more efficient in this section.

By using Transformation 16 in the optimal control problem, Eqs. (13)–(15) are transformed to the optimal control problems in Eqs. (17)–(19). First, we consider this approximation for $x(\cdot)$, which in terms of $T_i(t)$'s are the Chebyshev polynomials:

$$x_1(t) = \sum_{i=0}^{2} a_i T_i(t). \tag{21}$$

By using boundary conditions, we have:

$$a_0 = \frac{x^1 + x^0}{2} - a_2, \qquad a_1 = x^1 - \frac{x^1 + x^0}{2}. \tag{22}$$

Substituting Relations 22 into Eq. (21) yields:

$$\begin{aligned} x_1(t) &= a_2 T_2(t) + \left(x^1 - \frac{x^1 + x^0}{2}\right) T_1(t) \\ &\quad + \left(\frac{x^1 + x^0}{2} - a_2\right) T_0(t), \end{aligned} \tag{23}$$

and then $u(t)$ can be obtained from Eq. (17). Now, we obtain J as a function of a_2 by calculating:

$$\frac{t_1 - t_0}{2} \int_{-1}^{1} L\left(\frac{t_1 - t_0}{2} t + \frac{t_1 + t_0}{2}, x(t), u(t)\right) dt,$$

and refer to it as $J(a_2)$. Let a^* be the value which minimizes $J(a_2)$, then $J(a^*)$ is the solution of the optimal control problem in Eqs. (17)–(19). Also, the state and control variables can be calculated from a^* approximately.

In the next step, $x_2(t)$ is approximated as below:

$$x_2(t) = x_1(t) + \sum_{i=1}^{3} a_i T_i(t). \tag{24}$$

Then, using boundary conditions, we have:

$$a_1 = -a_3, \qquad a_2 = 0. \tag{25}$$

Substituting Relations 25 into Eq. (24) yields:

$$x_2(t) = x_1(t) + a_3 T_3(t) - a_3 T_1(t), \tag{26}$$

and then, $u(t)$ can be obtained from Eq. (17). Now, we obtain J as a function of a_3 by calculating:

$$\frac{t_1 - t_0}{2} \int_{-1}^{1} L\left(\frac{t_1 - t_0}{2} t + \frac{t_1 + t_0}{2}, x(t), u(t)\right) dt,$$

and refer to it as $J(a_3)$. If a^* is the value that minimizes $J(a_3)$, then $J(a^*)$ is the solution of the optimal control problem in Eqs. (17)–(19). Also, we can calculate state and control variables from a^* approximately.

By continuing this procedure, we obtain a favorable accuracy, for example in the $(n+1)$th step. The approximate solution is given by:

$$x_{n+1}(t) = x_n(t) + \sum_{i=n}^{n+2} a_i T_i(t). \tag{27}$$

As in previous steps, using boundary conditions, we have:

$$\begin{aligned} x_{n+1}(-1) &= x_n(-1) = x^0 \Rightarrow a_{n+2} T_{n+2}(-1) \\ &\quad + a_{n+1} T_{n+1}(-1) + a_n T_n(-1) = 0, \end{aligned} \tag{28}$$

and:

$$\begin{aligned} x_{n+1}(1) &= x_n(1) = x^1 \Rightarrow a_{n+2} T_{n+2}(1) \\ &\quad + a_{n+1} T_{n+1}(1) + a_n T_n(1) = 0. \end{aligned} \tag{29}$$

To calculate unknown coefficients, a_n, a_{n+1}, as a function of a_{n+2}, Eqs. (28) and (29) are solved simultaneously;

$$a_n = \frac{T_{n+1}(-1)T_{n+2}(1) - T_{n+1}(1)T_{n+2}(-1)}{T_n(-1)T_{n+1}(1) - T_n(1)T_{n+1}(-1)} a_{n+2}, \tag{30}$$

and:

$$a_{n+1} = \frac{T_n(-1)T_{n+2}(1) - T_n(1)T_{n+2}(-1)}{T_n(1)T_{n+1}(-1) - T_n(-1)T_{n+1}(1)} a_{n+2}. \tag{31}$$

Note that the denominator, as mentioned in Lemma 2, is not zero. So, Eqs. (27), (30) and (31) approximate the solution of the state variable as follows:

$$\begin{aligned} x_{n+1}(t) &= x_n(t) + a_{n+2} T_{n+2}(t) \\ &\quad + \frac{T_n(-1)T_{n+2}(1) - T_n(1)T_{n+2}(-1)}{T_n(1)T_{n+1}(-1) - T_n(-1)T_{n+1}(1)} a_{n+2} T_{n+1}(t) \\ &\quad + \frac{T_{n+1}(-1)T_{n+2}(1) - T_{n+1}(1)T_{n+2}(-1)}{T_n(-1)T_{n+1}(1) - T_n(1)T_{n+1}(-1)} a_{n+2} T_n(t), \end{aligned} \tag{32}$$

and then, $u(t)$ can be obtained from Eq. (17). Now, we obtain J as a function of a_{n+2} by calculating:

$$\frac{t_1 - t_0}{2} \int_{-1}^{1} L\left(\frac{t_1 - t_0}{2} t + \frac{t_1 + t_0}{2}, x(t), u(t)\right) dt,$$

and call it $J(a_{n+2})$. So, if a^* is the value that minimizes $J(a_{n+2})$, then $J(a^*)$ is the solution of the optimal control problem in Eqs. (17)–(19). State and control variables can be calculated from a^* approximately. The above results lead to the following algorithm, which obtains the optimal performance index, $J(\cdot)$.

Algorithm 2. *Object:* To obtain an optimal value for $J(\cdot)$.

Step 1. Choose an $\varepsilon > 0$.
Step 2. For $n = 1$, calculate:

$$\begin{aligned} x_1(t) &= a_2 T_2(t) + \left(x^1 - \frac{x^1 + x^0}{2}\right) T_1(t) \\ &\quad + \left(\frac{x^1 + x^0}{2} - a_2\right) T_0(t), \end{aligned}$$

and then, calculate $a_1^* \in \operatorname{Arg\,min}\{J(a) : a \in \mathbb{R}\}$ and set $\rho_1 = J(a_1^*)$.

Step 3. Set $n \to n+1$, and calculate:

$$x_{n+1}(t) = x_n(t) + a_{n+2}T_{n+2}(t) + \frac{T_n(-1)T_{n+2}(1) - T_n(1)T_{n+2}(-1)}{T_n(1)T_{n+1}(-1) - T_n(-1)T_{n+1}(1)} a_{n+2}T_{n+1}(t) + \frac{T_{n+1}(-1)T_{n+2}(1) - T_{n+1}(1)T_{n+2}(-1)}{T_n(-1)T_{n+1}(1) - T_n(1)T_{n+1}(-1)} a_{n+2}T_n(t).$$

Step 4. Calculate:

$$a^*_{n+1} \in \operatorname{Arg\,min}\{J(a) : a \in \mathbb{R}\}$$

and set:

$$\rho_{n+1} = J\left(a^*_{n+1}\right).$$

Step 5. If $|\rho_{n+1} - \rho_n| < \varepsilon$, then stop, otherwise return to Step 3.

In the following theorem, the convergence of the algorithm is proved.

Theorem 3. *If J has continuous first derivatives, then* $\lim_{n\to\infty} \rho_n = \alpha$, *where* $\alpha = \inf_Q J$.

Proof. If we define $\rho_n = \min_{a_n\in\mathbb{R}} J(a_n)$, then:

$$\rho_n = J(a^*_n),$$

such that:

$$a^*_n \in \operatorname{Arg\,min}\{J(a_n) : a_n \in \mathbb{R}\}.$$

Let:

$$x^*_n(t) \in \operatorname{Arg\,min}\{J(x(t)) : x(t) \in Q'_n\},$$

then:

$$J(x^*_n(t)) = \min_{x(t)\in Q'_n} J(x(t)),$$

in which Q'_n is a class of combinations of Chebyshev polynomials in t of degree n. It is obvious that $\rho_n = J(x^*_n(t))$. Furthermore, according to Corollary 1, we have:

$$\min_{x(t)\in Q'_{n+1}} J(x(t)) \le \min_{x(t)\in Q'_n} J(x(t)).$$

Thus, we will have $\rho_{n+1} \le \rho_n$ which means ρ_n is a non-increasing sequence. Now, according to Theorem 1, the proof is complete, that is:

$$\lim_{n\to\infty} \rho_n = \min_{x(t)\in Q} J(x(t)). \quad \square$$

In the next section, we apply the present algorithm to some engineering problems to show the efficiency and reliability of our method.

6. Numerical examples

To illustrate the efficiency of the presented algorithm, we consider the following examples. All problems considered have continuous optimal controls and can be solved analytically. This allows verification and validation of the method by comparison with results of exact solutions.

In [13], there exists a modified method, which searches for a single unknown coefficient similar to our work. As seen from Tables 1–3, our method converges more rapidly than in [13]. Besides, the problems of controlled linear and Duffing oscillators, which cannot be solved by the algorithm in [13], are solved here.

Example 1. In the following example, there is only one control function, $U(\tau)$, and only one state function, $X(\tau)$, that is concerned with minimization of:

Table 1: The optimal cost functional J for Example 1.

Iteration	Present method	Error	Mehne method [13]	Error
1	0.328598485	$3.3e^{-4}$	0.3333333333	$5.0e^{-3}$
2	0.328259338	$5.2e^{-7}$	0.3285984848	$3.4e^{-3}$
3	0.328258837	$1.6e^{-8}$	0.3284769571	$2.1e^{-4}$

Table 2: The optimal cost functional J for Example 2.

Iteration	Present method	Error	Mehne method [13]	Error
1	0.194298642	$1.3e^{-3}$	0.2513627360	$5.8e^{-2}$
2	0.192931607	$2.2e^{-5}$	0.194298642	$1.3e^{-3}$
3	0.192909776	$4.7e^{-7}$	0.193828723	$9.1e^{-4}$

Table 3: The optimal cost functional J for Example 3.

Iteration	Present method	Error	Mehne method [13]	Error
1	0.0840152601	$3.0e^{-5}$	0.05332622101	$3.0e^{-2}$
2	0.0840423344	$3.2e^{-6}$	0.0840152600	$3.0e^{-5}$
3	0.0840455804	$4.0e^{-8}$	0.08402496180	$2.0e^{-5}$

$$J = \int_0^1 (U(\tau)^2 + X(\tau)^2)d\tau, \quad \tau \in [0, 1], \tag{33}$$

subject to;

$$U(\tau) = \dot{X}(\tau), \tag{34}$$

with boundary conditions:

$$X(0) = 0, \qquad X(1) = \frac{1}{2}. \tag{35}$$

The analytical solution is [34]:

$$X(\tau) = \frac{e(e^\tau - e^{-\tau})}{2(e^2 - 1)}, \qquad U(\tau) = \frac{e(e^\tau + e^{-\tau})}{2(e^2 - 1)}. \tag{36}$$

In order to use the Chebyshev polynomials, we introduce the transformation $\tau = \frac{1}{2}t + \frac{1}{2}$. The optimal control problem in Eqs. (33)–(35) may then be restated as follows:
Minimize:

$$J = \frac{1}{2}\int_{-1}^1 (u(t)^2 + x(t)^2)dt, \quad t \in [-1, 1], \tag{37}$$

subject to:

$$u(t) = 2\dot{x}(t). \tag{38}$$

With:

$$x(-1) = 0, \qquad X(1) = \frac{1}{2}. \tag{39}$$

By using Step 2 of Algorithm 2, we consider an approximation of $x_1(t)$ to start with, as:

$$x_1(t) = a_2(2t^2 - 1) + \frac{t}{4} + \frac{1}{4} - a_2. \tag{40}$$

From Eq. (38), we have:

$$u(t) = 8a_2 t + \frac{1}{2}. \tag{41}$$

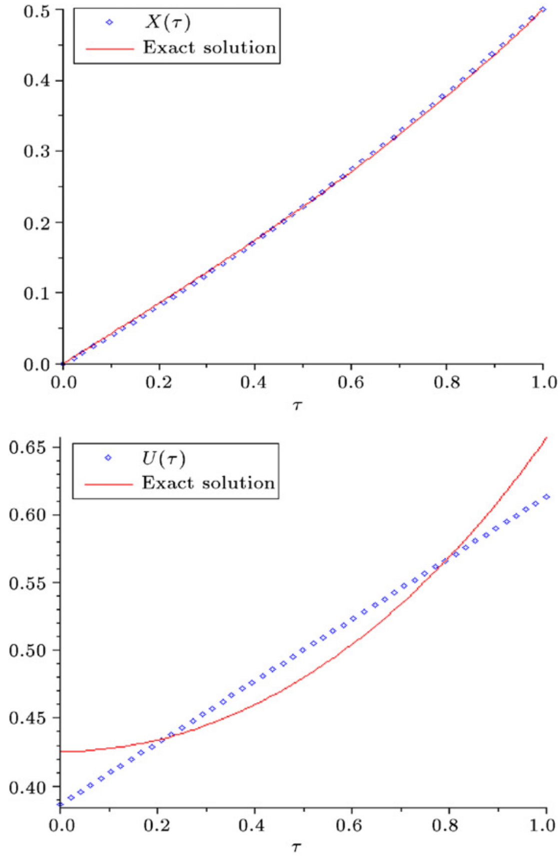

Figure 1: Solution of Example 1. The solution in the first iteration is compared with the actual analytical solution.

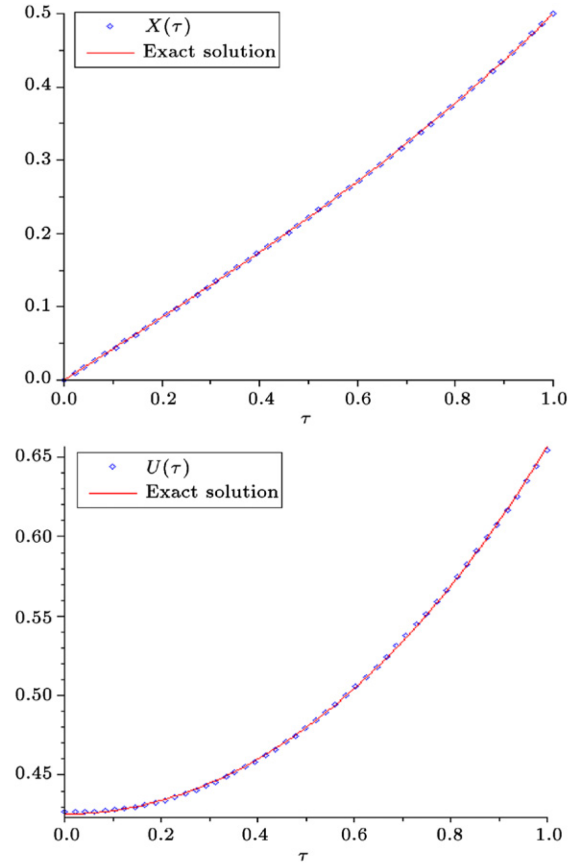

Figure 2: Solution of Example 1. The solution in the second iteration compared with the actual analytical solution.

Then, substituting Eqs. (40) and (41) into Eq. (37) gives:

$$J(a_2) = \frac{352}{15}a_2^2 - \frac{2}{3}a_2 + \frac{1}{3}. \tag{42}$$

Now, $a^* = \frac{5}{352}$ is the value which minimize J, then $J(a^*) = 0.328598485$ is the solution of the optimal control problem (37)–(39), and substituting a^* into Eqs. (40) and (41) and also the transformation $t = 2\tau - 1$, we can calculate state and control variables approximately as:

$$X_1(\tau) = \frac{5}{44}\tau^2 + \frac{17}{44}\tau, \tag{43}$$

and:

$$U(\tau) = \frac{5}{22}\tau + \frac{17}{44}. \tag{44}$$

The obtained solution and the analytical solution are plotted in Figure 1.

In the next step, $x_2(t)$ approximates the solution as follows:

$$x_2(t) = x_1(t) + \sum_{i=1}^{3} a_i T_i(t). \tag{45}$$

Now, the results of repeating the above procedure are shown in Figure 2.

The approximate solution for the performance index, as given in [34], is $J = 0.3282588215$. The optimal cost functional, J, obtained by the presented algorithm, is shown in Table 1.

Example 2 (*Problem Treated by El-Gindy et al. [24]*)**.** The objective is to find the optimal control $U(\tau)$, which minimizes:

$$J = \frac{1}{2}\int_0^1 (U(\tau)^2 + X(\tau)^2)d\tau, \quad \tau \in [0, 1], \tag{46}$$

when:

$$U(\tau) = \dot{X}(\tau) + X(\tau), \tag{47}$$

and:

$$X(0) = 1, \tag{48}$$

are satisfied. We have obtained the analytical solution by use of Pontryagin's maximum principle, which is:

$$X(\tau) = Ae^{\sqrt{2}\tau} + (1 - A)e^{-\sqrt{2}\tau},$$

$$U(\tau) = A\left(\sqrt{2}+1\right)e^{\sqrt{2}\tau} - (1 - A)\left(\sqrt{2}-1\right)e^{-\sqrt{2}\tau},$$

$$\begin{aligned} J = &+\frac{e^{-2\sqrt{2}}}{2}\left(\left(\sqrt{2}+1\right)\left(e^{4\sqrt{2}}-1\right)\right)A^2 \\ &+\frac{e^{-2\sqrt{2}}}{2}\left(\left(\sqrt{2}-1\right)\left(e^{2\sqrt{2}}-1\right)\right)(1-A)^2, \end{aligned} \tag{49}$$

where:

$$A = \frac{2\sqrt{2}-3}{-(e^{\sqrt{2}})^2 + 2\sqrt{2} - 3}. \tag{50}$$

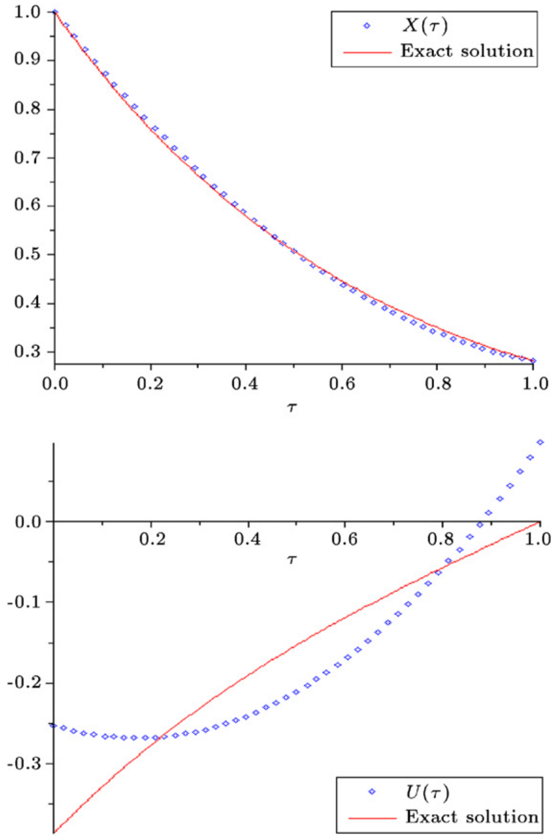

Figure 3: Solution of Example 2. The solution in the first iteration is compared with the actual analytical solution.

Transforming τ to time interval $[-1, 1]$, The problem is then redefined as:

Minimize

$$J = \frac{1}{4}\int_{-1}^{1}(u(t)^2 + x(t)^2)dt, \tag{51}$$

subject to

$$u(t) = 2\dot{x}(t) + x(t), \quad -1 \le t \le 1, \tag{52}$$

and;

$$x(-1) = x_0 = 1, \qquad x(1) = x_1 = 0.2819695348. \tag{53}$$

Note that to be able to use the presented algorithm, $X(1)$ needs to be calculated. Here, we have calculated $X(1)$ from Eq. (49).

By using Step 2 of Algorithm 2, we consider an approximation of $x_1(t)$ to start with, as:

$$x_1(t) = a_2(2t^2 - 1) - 0.3590t + 0.64109 - a_2, \tag{54}$$

from Eq. (52), we have:

$$u(t) = 2a_2t^2 + (8a_2 - 0.3590)t - 2a_2 - 0.0770. \tag{55}$$

Then, substituting Eqs. (54) and (55) into Eq. (51) gives:

$$J(a_2) = 12.8000a_2^2 - 1.7093a_2 + 0.2514. \tag{56}$$

Now, $a^* = 0.0668$ is the value that minimizes J, then $J(a^*) = 0.194298642$ is the solution of the optimal control problem 51–53, and substituting a^* into Eqs. (54) and (55) and also the

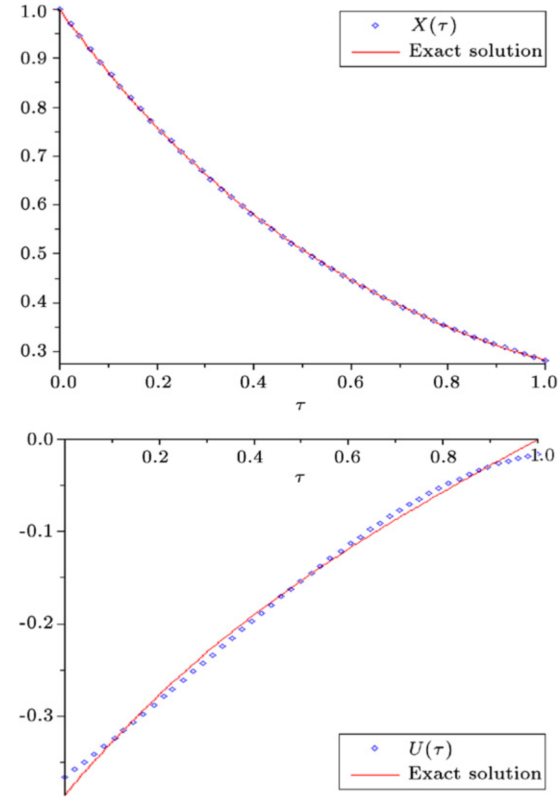

Figure 4: Solution of Example 2. The solution in the second iteration compared with the actual analytical solution.

transformation $t = 2\tau - 1$, we can calculate state and control variables approximately as:

$$X_1(\tau) = 0.5341\tau^2 - 1.2522\tau + 1, \tag{57}$$

and:

$$U(\tau) = 0.5342\tau^2 - 0.1839\tau - 0.2522. \tag{58}$$

The obtained solution and the analytical solution are plotted in Figure 3.

In the next step, $x_2(t)$ approximates the solution as follows:

$$x_2(t) = x_1(t) + \sum_{i=1}^{3} a_i T_i(t). \tag{59}$$

Now, the results of repeating the above procedure are shown in Figure 4.

The approximate solution for the performance index is $J = 0.1929092978$. The optimal cost functional, J, obtained by the presented algorithm is shown in Table 2.

Example 3. The following example [35] is concerned with minimization of:

$$J = \int_0^1 \left(X(\tau) - \frac{1}{2}U(\tau)^2\right) d\tau, \quad \tau \in [0, 1], \tag{60}$$

subject to:

$$U(\tau) = \dot{X}(\tau) + X(\tau), \tag{61}$$

with boundary conditions:

$$X(0) = 0, \qquad X(1) = \frac{1}{2}\left(1 - \frac{1}{e}\right)^2, \tag{62}$$

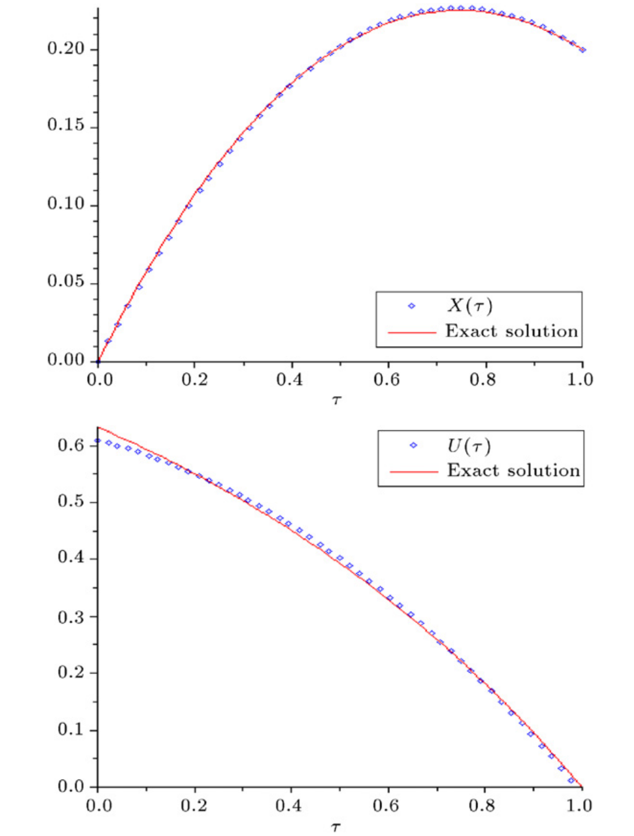

Figure 5: Solution of Example 3. The solution of the first iterative compared with the actual analytical solution.

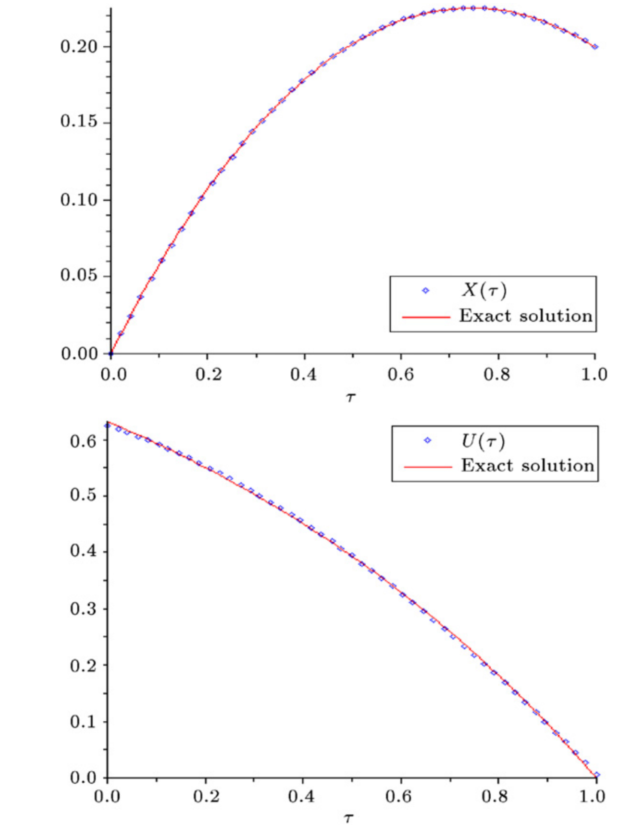

Figure 6: Solution of Example 3. The solution in the second iteration compared with the actual analytical solution.

where the analytical solution is:

$$X(\tau) = 1 - \frac{1}{2}e^{\tau-1} + \left(\frac{1}{2e} - 1\right)e^{-\tau},$$

$$U(\tau) = 1 - e^{\tau-1}. \tag{63}$$

We can calculate state and control variables approximately as:

$$X_1(\tau) = -0.4091\tau^2 + 0.6089\tau, \tag{64}$$

and:

$$U(\tau) = -0.4091\tau^2 - 0.2095\tau + 0.6089. \tag{65}$$

The obtained solution and the analytical solution are plotted in Figure 5.

Now, the results of repeating the above procedure are shown in Figure 6.

The approximate solution for the performance index, as given in [32], is $J = 0.08404562020$. The optimal cost functional, J, obtained by the presented algorithm is shown in Table 3.

7. The controlled linear oscillator

We will consider the optimal control of a linear oscillator governed by the differential equation:

$$U(\tau) = \ddot{X}(\tau) + \omega^2 X(\tau), \quad \tau \in [-T, 0], \tag{66}$$

in which dot $(\cdot)$ means differentiation, with respect to τ, and T is specified. Eq. (66) is equivalent to the dynamic state equations:

$$\dot{X}_1(\tau) = X_2(\tau),$$

$$\dot{X}_2(\tau) = -\omega^2 X_1(\tau) + U(\tau), \tag{67}$$

with the boundary conditions:

$$X_1(-T) = X_{10}, \qquad X_2(-T) = X_{20},$$

$$X_1(0) = 0, \qquad X_2(0) = 0. \tag{68}$$

It is desired to control the state of this plant, such that the performance index,

$$J = \frac{1}{2}\int_{-T}^{0} U^2(\tau)d\tau, \tag{69}$$

is minimized over all admissible control functions, $U(\tau)$.

Pontryagin's maximum principle method [3], applied to this optimal control problem, yields the following exact analytical solution [36]:

$$X_1(\tau) = \frac{1}{2\omega^2}\left[A\omega\tau \sin\omega\tau + B(\sin\omega\tau - \omega\tau\cos\omega\tau)\right],$$

$$X_2(\tau) = \frac{1}{2\omega}\left[A(\omega\tau\sin\omega\tau + \omega\tau\cos\omega\tau) + B\omega\tau\sin\omega\tau\right],$$

$$U(\tau) = A\cos\omega\tau + B\sin\omega\tau,$$

$$J = \frac{1}{8\omega}\left[2\omega T\left(A^2 + B^2\right) + \left(A^2 - B^2\right)\sin 2\omega T - 4AB\sin^2\omega T\right], \tag{70}$$

where:

$$A = \frac{2\omega[x_{10}\omega^2 T\sin\omega T - x_{20}(\omega T\cos\omega T - \sin\omega T)]}{\omega^2T^2 - \sin^2\omega T},$$

$$B = \frac{2\omega^2[x_{20}T\sin\omega T + x_{10}(\omega T\cos\omega T + \sin\omega T)]}{\omega^2T^2 - \sin^2\omega T}. \tag{71}$$

7.1. Solution of the problem using Algorithm 2

The optimal control problem described in Eqs. (66)–(69) can be restated as follows:

Minimize

$$J = \frac{T}{4}\int_{-1}^{1} u^2(t)dt, \tag{72}$$

subject to:

$$u(t) = \omega^2 x(t) + \frac{4}{T^2}\omega^2 \ddot{x}(t), \quad t \in [-1, 1], \tag{73}$$

with:

$$x(-1) = x_{-1}, \qquad \dot{x}(-1) = \dot{x}_{-1},$$
$$x(1) = 0, \qquad \dot{x}(1) = 0. \tag{74}$$

We consider this approximation of $x(\cdot)$ to start with:

$$x_1(t) = \sum_{i=0}^{4} a_i T_i(t). \tag{75}$$

Using boundary conditions (74), we have:

$$\begin{cases} a_0 - a_1 + a_2 - a_3 + a_4 = x_{-1} \\ a_0 + a_1 + a_2 + a_3 + a_4 = 0 \\ a_1 + 4a_2 + 9a_3 - 16a_4 = \dot{x}_{-1} \\ a_1 + 4a_2 + 9a_3 + 16a_4 = 0. \end{cases} \tag{76}$$

Then, a_0, a_1, a_2, a_3 are obtained as a function of a_4 by solving the linear system of equations given by:

$$a_0 = \frac{1}{2}x_{-1} + \frac{1}{8}\dot{x}_{-1} + 3a_4,$$
$$a_1 = -\frac{9}{16}x_{-1} - \frac{1}{16}\dot{x}_{-1},$$
$$a_2 = -4a_4 - \frac{1}{8}\dot{x}_{-1} + 3a_4,$$
$$a_3 = \frac{1}{16}x_{-1} + \frac{1}{16}\dot{x}_{-1}. \tag{77}$$

Now, substituting Eq. (77) into (75) yields:

$$x_1(t) = 8a_4t^4 + \frac{1}{4}\dot{x}_{-1}t^3 - \left(16a_4 + \frac{1}{4}\dot{x}_{-1}\right)t^2 - \left(\frac{3}{4}x_{-1} + \frac{1}{4}\dot{x}_{-1}\right)t + \left(8a_4 + \frac{1}{2}x_{-1} + \frac{1}{4}\dot{x}_{-1}\right), \tag{78}$$

and then $u(t)$ is obtained from Eq. (73)

$$\begin{aligned} u(t) = {} & \frac{1}{4T^2}(32T^2\omega^2 a_4 t^4 + T^2\omega^2(\dot{x}_{-1} + x_{-1})t^3 \\ & - T^2\omega^2(\dot{x}_{-1} + x_{-1})t^2 - T^2\omega^2(\dot{x}_{-1} + 3x_{-1}) \\ & + T^2\omega^2(32a_4 + \dot{x}_{-1} + 2x_{-1}) + 1536a_4t^2 \\ & + 24(x_{-1} + \dot{x}_{-1})t - 512a_4 - 8\dot{x}_{-1}). \end{aligned} \tag{79}$$

Now, we obtain J as a function of a_4 by calculating $\frac{T}{2}\int_{-1}^{1} u^2(t)dt$, and denote it by $J(a_4)$. So, the value which minimizes $J(a_4)$ is given by:

$$a^* = -\frac{3}{256}\frac{T^2\omega^2(7T^2\omega^2 + 3T^2\omega^2 x_{-1} - 56\dot{x}_{-1})}{T^4\omega^4 - 24T^2\omega^2 + 504}. \tag{80}$$

$J(a^*)$ is the solution of the optimal control problem (Eqs. (72)–(74)). Also, we can calculate state and control variables from a^* approximately. Now, we report the Chebyshev approximation of the state and control variables of the controlled linear oscillator problem, with the following choice of numerical values of parameters in a standard case:

$$\omega = 1, \qquad T = 2, \qquad x_{-1} = 0.5, \qquad \dot{x}_{-1} = -0.5. \tag{81}$$

Substituting the values in Eq. (81) into Eqs. (78) and (79) yields:

$$x_1(t) = 8a_4t^4 - \left(16a_4 - \frac{1}{8}\right)t^2 - \frac{1}{4}t + 8a_4 + \frac{1}{8}, \tag{82}$$

and:

$$u(t) = 8a_4t^4 + \left(80a_4 + \frac{1}{8}\right)t^2 - \frac{1}{4}t - 24a_4 + \frac{3}{8}. \tag{83}$$

Also, by Eq. (72), we have:

$$J(a_4) = \frac{217088}{315}a_4^2 + \frac{192}{35}a_4 + \frac{47}{240}, \tag{84}$$

where $a^* = -\frac{27}{6784}$ is the value which minimizes $J(a_4)$, and $J(a^*) = 0.184916891$ is the solution of the optimal control problem (Eqs. (72)–(74)). By substituting a^* into Eqs. (82) and (83) and also using the transformation, $t = \frac{2}{T}\tau + 1$, we can calculate state and control variables approximately as:

$$X_1(\tau) = -0.0318\tau^4 - 0.1274\tau^3 - 0.0024\tau^2, \tag{85}$$

and:

$$U(\tau) = -0.0318\tau^4 - 0.1274\tau^3 - 0.3844\tau^2 - 0.7642\tau - 0.0047. \tag{86}$$

The solution obtained and the analytical solutions are plotted in Figure 7.

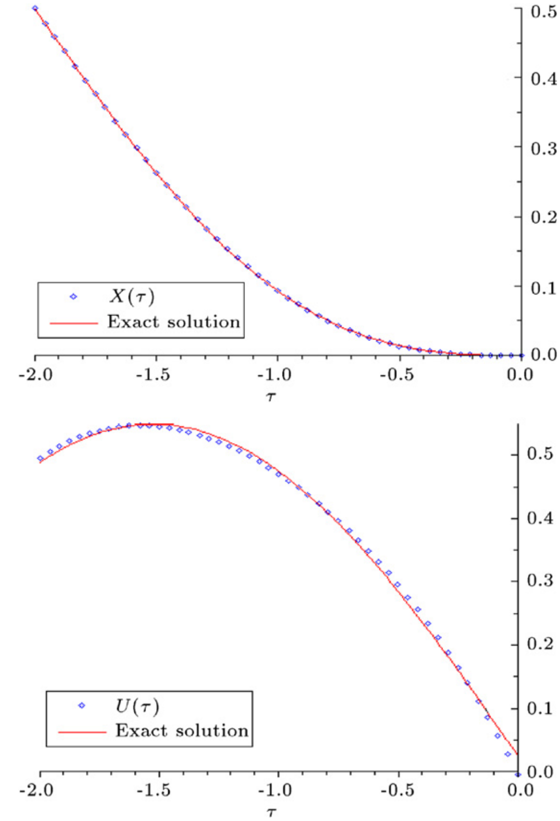

Figure 7: Solution of the controlled linear oscillator problem. The solution in the first iteration compared with the actual analytical solution.

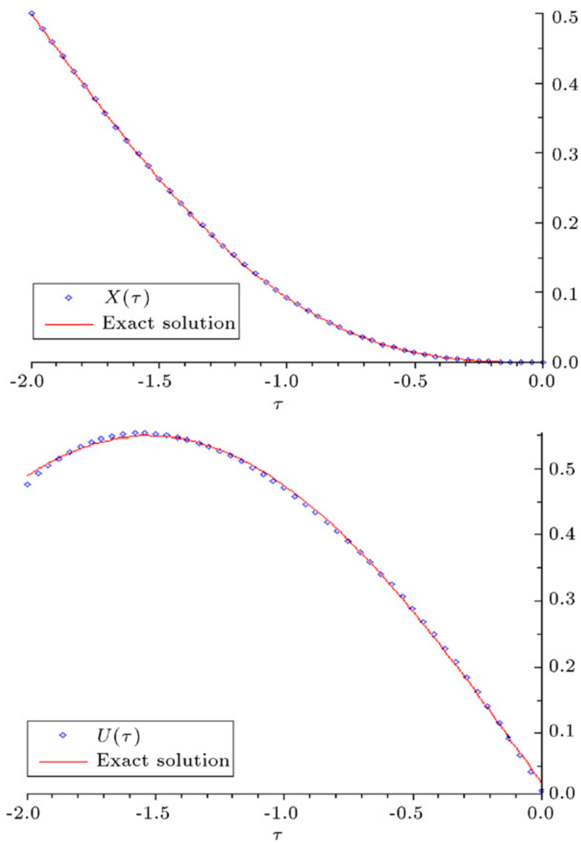

Figure 8: Solution of the controlled linear oscillator problem. The solution in the second iteration compared with the actual analytical solution.

Table 4: The optimal cost functional J for the controlled linear oscillator problem.

Iteration	Present method	Error
1	0.184916891	$0.5e^{-4}$
2	0.184873530	$0.14e^{-4}$
3	0.184858576	$3.4e^{-8}$

Now, the results of the above procedure repeated once again are shown in Figure 8.

The approximate solution for the performance index, as given in [36], is $J = 0.184858542$. The optimal cost functional, J, obtained by the presented algorithm, is shown in Table 4.

7.2. The controlled Duffing oscillator

Let us now investigate the optimal control of the Duffing oscillator, described by the nonlinear differential equation:

$$U(\tau) = \ddot{X}(\tau) + \omega^2 X(\tau) + \varepsilon X^3(\tau), \quad \tau \in [-T, 0]. \tag{87}$$

Subject to boundary conditions and with the performance index pointed out, as in the previously linear case, Eq. (87) is replaced by:

$$u(t) = \frac{4}{T^2}\ddot{x}(t) + \omega^2 x(t) + \varepsilon x^3(t), \quad t \in [-1, 1]. \tag{88}$$

The exact solution in this case is not known.

Table 5 lists the optimal values of the cost functional J for various values of ε in three iterations.

Table 5: The optimal cost functional J for Duffing oscillator problem for various values of ε.

	Present method		
	$\varepsilon = 0.15$	$\varepsilon = 0.5$	$\varepsilon = 0.75$
1	0.187529215	0.193708049	0.198192304
2	0.187456249	0.193534882	0.197920110
3	0.187444872	0.193530147	0.197918461

The approximate solution for the performance index, as given in [24,37], is $J = 0.187444856$, with $\varepsilon = 0.15$ and $J = 0.19353033$ for $\varepsilon = 0.5$, also, $J = 0.19791863$ for $\varepsilon = 0.75$.

8. Conclusion

In this paper, a new computational algorithm for minimizing the performance index was obtained by utilizing Chebyshev polynomials. This algorithm provides a simple way to adjust and obtain an optimal control that can easily be applied to complex problems as well. One advantage of this method is the use of a computational algorithm with fast convergence. This algorithm, which is a modification of the algorithm in [13], can be used to approximate control and state variables as a function of time. Some examples were solved by this algorithm, and the results show that the presented algorithm is more powerful and sufficient than that in previous work, requiring less computational work and, hence, creating a significant reduction in computational costs, which is an important factor when choosing a method in engineering applications. The suggested algorithm was then applied to control the Duffing oscillator problem. The results obtained demonstrate and emphasize the reliability and efficiency of the proposed algorithm.

References

[1] Fleming, W.H. and Rishel, R.W., *Deterministic and Stochastic Optimal Control*, Springer-Verlag, New York (1975).
[2] Bellman, R., *Dynamic Programming*, University Press, Princeton, NJ (1957).
[3] Pontryagin, L.S. "The mathematical theory of optimal processes", In *Interscience*, John Wiley & Sons (1962).
[4] Rubio, J.E., *Control and Optimization: The Linear Treatment of Non-linear Problems*, Manchester University Press, Manchester (1986).
[5] Pinch, E.R., *Optimal Control and the Calculus of Variations*, Oxford University Press, London (1993).
[6] Delavarkhalafi, A. "Computation of switching surfaces using Grobner basis", *Lect. Ser. Comput. Comput. Sci.*, 4, pp. 143–145 (2005).
[7] Karbassi, S.M. and Tehrani, H.A. "Parameterizations of the state feedback controllers for linear multivariable systems", *Comput. Math. Appl.*, 44, pp. 1057–1065 (2002).
[8] Ahmed, N.U., *Elements of Finite-Dimensional Systems and Control Theory*, Longman Scientific and Technical, England (1988).
[9] Craven, B.D., *Control and Optimization*, Chapman & Hall, London (1995).
[10] Miele, A. "Control and Dynamic Systems: Advances in Theory and Applications", in: C.T. Leondes (Ed.), *Gradient algorithms for the optimization of dynamic systems*, 16, pp. 1–52, Academic Press, New York (1980).
[11] Sakawa, Y. and Shindo, Y. "On the global convergence of an algorithm for optimal control", *IEEE Trans. Automatic Control AC-25*, pp. 1149–1153 (1980).
[12] Teo, K.L., Goh, C.J. and Wong, K.H., *A Unified Computational Approach to Optimal Control Problems*, Longman Scientific and Technical, England (1991).
[13] Mehne, H.H. and Hashemi Borzabadi, A. "A numerical method for solving optimal control problems using state parametrization", *Numer. Algorithms*, 42, pp. 165–169 (2006).
[14] Ashoori, A., Moshiri, B., Ramezani, A., Bakhtiari, M.R. and Khaki-Sedigh, A. "Optimal control of a nonlinear fed-batch fermentation process using model predictive approach", *J. Process Control*, 19, pp. 1162–1173 (2009).
[15] Teo, K.L., Jennings, L.S., Lee, H.W.J. and Rehbock, V. "The control parametrization enhancing transform for constrained optimal control problems", *J. Austral. Math. Soc. Ser. B*, 40, pp. 314–335 (1999).

[16] Lee, H.W.J., Teo, K.L., Jennings, L.S., Lee, H.W.J. and Rehbock, V. "Control parametrization enhancing technique for time optimal control problem", *Dynamical Syst. Appl.*, 6, pp. 243–261 (1977).
[17] Teo, K.L. "Control parametrization enhancing transform to optimal control problems", *Nonlinear Anal.*, 63, pp. e2223–e2236 (2005).
[18] Vlassenbroeck, J. "A Chebyshev polynomial method for optimal control with state constraints", *Automatica*, 24(4), pp. 499–506 (1988).
[19] Jaddu, H.M. "Numerical methods for solving optimal control problems using Chebyshev polynomials", Ph.D. Thesis, School of Information Science, Japan Advanced Institute of Science and Technology (1998).
[20] Jaddu, H.M. "Direct solution of nonlinear optimal control problems using quasilinearization and Chebyshev polynomials", *J. Franklin Inst.*, 339(4–5), pp. 479–498 (2002).
[21] Lee, Y.C.E., Fung, E.H.K. and Lee, H.W.J. "Control parametrization enhancing technique and simulation on the design of a flexible rotating beam", *J. Optim. Theory Appl.*, 136, pp. 247–259 (2008).
[22] Van Dooren, R. and Vlassenbroeck, J. "Chebyshev series solution of the controlled Duffing oscillator", *J. Comput. Phys.*, 47, pp. 321–329 (1982).
[23] El-Gendi, S.E. "Chebyshev solution of differential, integral, and integro–differential equations", *Comput. J.*, 12, pp. 282–287 (1969).
[24] El-Gindy, T.M., El-Hawary, H.M., Salim, M.S. and El-Kady, M. "A Chebyshev approximation for solving optimal control problems", *Comput. Math. Appl.*, 29, pp. 35–45 (1995).
[25] El-Kady, M., Elsayed, M.E. and Elbarbary, "A Chebyshev expansion method for solving nonlinear optimal control problems", *Appl. Math. Comput.*, 129, pp. 171–182 (2002).
[26] Frick, P.A. and Stech, D.J. "Solution of optimal control problems on a parallel machine using the Epsilon method", *Optim. Control Appl. Methods*, 16, pp. 1–17 (1995).
[27] Jennings, L.S., Fisher, M.E., Teo, K.L. and Goh, C.J. "MISER3 optimal control software, Version 3: Theory and user manual". http://www.maths.uwa.edu.au/~les/miser3.3.html.
[28] Jennings, L.S., Teo, K.L. and Goh, C.J. "MISER3.2 optimal control software: theory and user manual", Department of Mathematics, The University of Western Australia, Australia (1997). http://www.cado.maths.uwa.edu.au.
[29] Teo, K.L., Goh, C.L. and Wong, K.H., *A Unified Computational Approach to Optimal Control Problems*, Longman Scientific and Technical, London (1991).
[30] Stoer, J. and Bulirsch, R., *Introduction to Numerical Analysis*, 2nd Edn., Springer-Verlag, New York, Translated by R. Bartels, W. Gautschi, C. Witzgall (1993).
[31] Mason, J.C. and Handscomb, D.C., *Chebyshev Polynomials*, A CRC Press Company, United States of America (2003).
[32] Rivlin, T.J., *Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory*, John Wiley, New York (2nd Edn., of Rivlin (1974)) (1990).
[33] Rudin, W., *Principles of Mathematical Analysis*, 3rd Edn., McGraw-Hill, New York (1976).
[34] Pantelev, A.B., Bortakovski, A.C. and Letova, T.A., *Some Issues and Examples in Optimal Control*, MAI Press, Moscow (in Russian) (1996).
[35] Naevdal, E. "Solving continuous-time optimal-control problems with a spreadsheet", *J. Econom. Educ.*, 34 (2), pp. 99–122.
[36] Van Dooren, R. and Vlassenbroeck, J. "A Chebyshev technique for solving nonlinear optimal control problems", *IEEE Trans. Automat. Control*, 33(4), pp. 333–339 (1988).
[37] Kelley, H.J. "Methods of gradients", In *Optimization Techniques*, G. Leitmann, Ed., pp. 205–254, Academic Press, London (1962).

Behzad Kafash received a B.S. degree, in 2004, from Payame Noor University, and an M.S. degree, in 2008, from Yazd University, in Applied Mathematics. He is currently working towards a Ph.D. degree in Stochastic Optimal Control at Yazd University. His research interests are nonlinear, stochastic control and financial mathematics.

Ali Delavarkhalafi is Assistant Professor at Yazd University. He received a B.S. degree in 1989, an M.S. degree, in 1991, from Sistan and Baluchistan University, and a Ph.D. degree in Optimal Control from Moscow State University, in 2002. His research interests are stochastic optimal control, differential games and financial mathematics.

Seyed-Mehdi Karbassi received a B.S. degree in 1969, an M.S. degree, in 1970, from Newcastle University, UK, and a Ph.D. degree in Control Engineering from Salford University, UK, in 1978. He is currently a professor at the Islamic Azad University, Yazd Branch, Iran. His research interests are nonlinear, robust and intelligent control.