Automatica 47 (2011) 797–803

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

On H_∞ and H_2 performance regions of multi-agent systems[☆]

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ARTICLE INFO

Article history:
Available online 23 February 2011

Keywords:
Multi-agent system
Distributed control
H_2 control
H_∞ control
Performance region
Performance limit

ABSTRACT

This paper addresses the distributed H_2 and H_∞ control problems for multi-agent systems with linear or linearized dynamics. An undirected multigraph with loops is used to represent the communication topology of a multi-agent network. A distributed controller is designed, based on the relative states of neighboring agents and a subset of absolute states of the networked agents. The notions of H_∞ and H_2 performance regions are introduced and analyzed, respectively. A necessary and sufficient condition for the existence of a controller yielding an unbounded H_∞ performance region is derived. A multi-step procedure for suboptimal H_∞ controller synthesis is presented. It is also shown that the H_∞ performance limit of the network under the distributed controller is equal to the minimal H_∞ norm of a single agent achieved by using the state feedback controller. It is finally shown that, contrarily to the H_∞ case, the H_2 performance limit scales with the number of agents in the network.

1. Introduction

In recent years, the coordination control problem of multi-agent systems has received increasing attention from scientific especially systems and control communities, for its broad applications in various fields such as satellite formation flying, cooperative unmanned air vehicles, sensor networks, and air traffic control, to name just a few. Due to the spatial distribution of actuators and limited sensing capability of sensors, it is considered too expensive or even infeasible in practice to implement centralized controllers. Thus, distributed control appears to be a promising tool for multi-agent systems.

Formation control of autonomous vehicles was considered in Fax and Murray (2004), where a Nyquist-like criterion was derived. In Gupta, Hassibi, and Murray (2005), it was concerned with the synthesis problem of a linear quadratic regulator (LQR) controller subject to certain particular vector space constraints, and distributed LQR control of a set of identical decoupled dynamical systems where the performance index couples the behavior of the systems was discussed in Borrelli and Keviczky (2008). A theoretical explanation was provided in Jadbabaie, Lin, and Morse (2003) for the behavior observed in the Vicsek model Vicsek, Czirók, Ben-Jacob, Cohen, and Shochet (1995). In Olfati-Saber and Murray (2004), a general framework of the consensus problem for networks of dynamic agents with fixed or switching topologies and communication time-delays was established. The conditions derived in Jadbabaie et al. (2003) and Olfati-Saber and Murray (2004) were further relaxed in Ren and Beard (2005). Also, in Hong, Chen, and Bushnell (2008); Hong, Hu, and Gao (2006), it was considered about tracking control for multi-agent consensus with an active leader, where a local controller was designed together with a neighbor-based state-estimation rule. A distributed algorithm was proposed in Cortés (2008) to asymptotically achieve consensus in finite time. The notion of consensus region was introduced in Li, Duan, Chen, and Huang (2010) as a basis for a multi-step consensus protocol design algorithm. Flocking algorithms were investigated in Olfati-Saber (2006), Su, Wang, and Lin (2009) and Tanner, Jadbabaie, and Pappas (2007) for a group of autonomous agents. A decomposition approach was proposed in Massioni and Verhaegen (2009) to solve the distributed H_2 and H_∞ control of identical coupled linear systems. Last but not least, stability analysis and decentralized control problems for linear and sector-nonlinear complex dynamical networks were studied in Duan, Wang, Chen, and Huang (2008). More investigation reports can be found from the references of the aforementioned papers.

This paper considers distributed H_∞ and H_2 control of a multi-agent system with linear or linearized dynamics. An undirected multigraph with loops is used to model the communication topology of the multi-agent network. Contrary to the assumption

[☆] This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Hideaki Ishii under the direction of Editor Ian R. Petersen.

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doi:10.1016/j.automatica.2011.01.054

that every system has access to its own state or output (e.g., in Borrelli & Keviczky, 2008; Fax & Murray, 2004; Massioni & Verhaegen, 2009), it is supposed in this paper that only a subset of agents have complete information about their own states. Thus, the cooperation in terms of information exchanges among neighboring agents becomes vital to achieve the given control goal. The distributed controller proposed here is based on the relative states of neighboring agents and a subset of absolute states of the networked agents. A distinct feature of this paper is that by introducing a positive scalar, i.e., the coupling strength, the novel notions of H_∞ and H_2 performance regions are introduced and characterized. The H_∞ and H_2 performance regions, which can be regarded as an extension of the consensus region introduced in Li et al. (2010) to evaluate the performance of a multi-agent network subject to external disturbances, pave a new way for distributed controller synthesis, differentiating the present paper from related works Li, Duan, and Huang (2009) and Massioni and Verhaegen (2009). It will be pointed out through several examples that the H_∞ and H_2 performance regions can serve as a measure for the robustness of the distributed controller with respect to the communication topology of the multi-agent network.

The distributed H_∞ control problem of the multi-agent network is converted to the H_∞ control problem of a set of independent systems of the same dimension as a single agent. A necessary and sufficient condition for the existence of a distributed controller yielding an unbounded H_∞ performance region is derived, based on which a multi-step procedure for H_∞ controller synthesis is further presented, which maintains a favorable decoupling property. It is shown that the H_∞ performance limit of the network under the distributed controller is equal to the minimal H_∞ norm of an isolated agent achieved by using the state feedback controller, independent of the communication topology as long as it is connected. To the best of the authors' knowledge, it is the first time to obtain the exact H_∞ performance limit of distributed control for linear multi-agent systems, although with the limitation of not being able to weight the control effort in the performance output, while the results in related works Li et al. (2009) and Massioni and Verhaegen (2009) allow putting a penalty on the control effort but they are conservative. Contrary to the H_∞ case, using H_2 performance region to characterize the H_2 performance of a multi-agent network introduces certain conservatism, and the H_2 performance limit of the multi-agent network scales with the number of agents, implying the inherent difference between the H_2 and H_∞ norms used for distributed control of multi-agent networks. Finally, it is worth mentioning that the common distributed controller adopted here for a network of identical agents may set a limit on the H_∞ and H_2 performances. Yet, whether or not different controllers for identical agents can improve these performances is an interesting topic for future studies.

The rest of this paper is organized as follows. Some useful results of the graph theory are introduced and the problem is formulated in Section 2. The H_∞ performance region is analyzed and the proposed controller is designed in Section 3. The H_2 performance region is considered in Section 4. Section 5 concludes the paper.

Throughout this paper, the following notations will be used: let $\mathbf{R}^{n\times n}$ be the set of $n\times n$ real matrices. $\mathbf{R}_+$ denotes the set of positive real numbers. Matrices, if not explicitly stated, are assumed to have compatible dimensions. The superscript T means the transpose for real matrices. I_N represents the identity matrix of dimension N. Denote by $\mathbf{1}$ the column vector with all entries equal to one. $\mathcal{L}_2[0,\infty)$ denotes the space of square integrable vector functions over $[0,\infty)$. For real symmetric matrices X and Y, $X > Y$ means that matrix $X-Y$ is positive definite. $A\otimes B$ denotes the Kronecker product of matrices A and B. $\mathrm{diag}(A_1,\ldots,A_n)$ represents a block-diagonal matrix with matrices A_i, $i=1,\ldots,n$, on its diagonal. For a square matrix A, $\bar{\sigma}(A)$ denotes its maximal singular value and $\mathrm{tr}(A)$ denotes its trace. A matrix is Hurwitz (or stable) if all of its eigenvalues have negative real parts.

2. Preliminaries

2.1. Graph theory

An undirected graph $\mathcal{G}$ is a pair $(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{1,\ldots,p\}$ is the set of nodes and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the set of unordered pairs of nodes, called edges. Two nodes i,j are adjacent, or neighboring, if (i,j) is an edge of graph $\mathcal{G}$. The edges in the form of (i,i) are called loops. A graph with loops is called a multigraph, otherwise is a simple graph (Diestel, 1997). A path on $\mathcal{G}$ from node i_1 to node i_l is a sequence of ordered edges of the form (i_k,i_{k+1}), $k=1,\ldots,l-1$. An undirected graph is connected if there exists a path between every pair of distinct nodes, otherwise is disconnected.

The adjacency matrix $\mathcal{A}\in\mathbf{R}^{p\times p}$ of graph $\mathcal{G}$ is defined by $a_{ii}=1$ if node i has a loop but 0 otherwise, and $a_{ij}=a_{ji}=1$ if $(i,j)\in\mathcal{E}$ but 0 otherwise. The Laplacian matrix $\mathcal{L}\in\mathbf{R}^{p\times p}$ is defined as $\mathcal{L}_{ii}=\sum_{j=1}^{p}a_{ij}$, $\mathcal{L}_{ij}=-a_{ij}$ for $i\neq j$. To avoid ambiguity, denote by $\mathcal{L}_m$ the Laplacian matrix of a multigraph and by $\mathcal{L}_s$ the Laplacian matrix of a simple graph. For an undirected graph, both its adjacency matrix and its Laplacian matrix are symmetric.

Lemma 1 (Ren & Beard, 2005). *For a simple graph, 0 is an eigenvalue of $\mathcal{L}_s$ with $\mathbf{1}^T$ as the corresponding right eigenvector and all the nonzero eigenvalues have positive real parts. Furthermore, 0 is a simple eigenvalue of $\mathcal{L}_s$ if and only if the graph is connected.*

Lemma 2. *For a multigraph with at least one loop, the Laplacian matrix $\mathcal{L}_m$ is positive definite, if the graph is connected.*

Proof. By the definition of $\mathcal{L}_m$, it can be written as $\mathcal{L}_m=\widetilde{\mathcal{L}}_s+\tilde{A}$, where $\widetilde{\mathcal{L}}_s$ is the Laplacian matrix of the graph with all loops being removed and $\tilde{A}=\mathrm{diag}(a_{11},\ldots,a_{pp})$ having at least one diagonal item being positive. Thus, the positive definiteness of $\mathcal{L}_m$ associated with a connected graph follows directly from either Lemma 1 in Chen, Liu, and Lu (2007) or Lemma 3 in Hong et al. (2006). □

2.2. Problem formulation

Consider a network of N identical agents with linear or linearized dynamics, described by

$$\begin{aligned}\dot{x}_i&=Ax_i+B_1u_i+D_1\omega_i,\\ z_i&=Cx_i+D_2\omega_i,\quad i=1,2,\ldots,N,\end{aligned}\tag{1}$$

where $x_i\in\mathbf{R}^n$ is the state of the ith agent, $u_i\in\mathbf{R}^p$ is the control input, $\omega_i\in\mathcal{L}_2^{m_1}[0,\infty)$ is the exogenous input including plant disturbances, measurement noise, etc., $z_i\in\mathbf{R}^{m_2}$ denotes the performance variable, and A, B_1, C, D_1, D_2 are constant matrices with compatible dimensions.

The communication topology among the N agents is represented by an undirected multigraph $\mathcal{G}$ consisting of the node set $\mathcal{V}=\{1,\ldots,N\}$ and the edge set $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$. A loop (i,i) means that agent i knows its own state, and an edge (i,j) $(i\neq j)$ means that agents i and j can obtain information from each other.

Differing from the assumption in Borrelli and Keviczky (2008), Fax and Murray (2004) and Massioni and Verhaegen (2009) that the local state is available to every agent, it is supposed here that only a subset of agents know their own states. In this case, it is critical for neighboring agents to exchange information in order to achieve a given control goal. Without loss of generality, assume that the first q $(q\ll N)$ agents have access to their state information, i.e., there are loops around the first q nodes in graph $\mathcal{G}$.

A distributed control law is proposed here as

$$u_i = cK\left(\sum_{j=1}^{N} a_{ij}(x_i - x_j) + a_{ii}x_i\right), \tag{2}$$

where $K \in \mathbf{R}^{p\times n}$ are feedback gain matrices to be determined, $c > 0$ denotes the coupling strength, $\mathcal{A} = (a_{ij})_{N\times N}$ is the adjacency matrix of graph $\mathcal{G}$ with $a_{ii} = 1$ for $i = 1, \ldots, q$, and $a_{ii} = 0$ for $i = q+1, \ldots, N$.

Let $x = [x_1^T, \ldots, x_N^T]^T$, $\omega = [\omega_1^T, \ldots, \omega_N^T]^T$, and $z = [z_1^T, \ldots, z_N^T]^T$. Then, the closed-loop system resulting from (1) and (2) can be written as

$$\begin{aligned}\dot{x} &= (I_N \otimes A + c\mathcal{L}_m \otimes B_1K)x + (I_N \otimes D_1)\omega,\\ z &= (I_N \otimes C)x + (I_N \otimes D_2)\omega,\end{aligned} \tag{3}$$

where $\mathcal{L}_m$ is the Laplacian matrix associated with graph $\mathcal{G}$. Denote by $T_{\omega z}$ the transfer function matrix from ω to z of system (3).

The suboptimal H_∞ control problem for system (3) is stated as follows: for a given allowable $\gamma > 0$, find a distributed controller (2) such that (i) system (3) is asymptotically stable; (ii) $\|T_{\omega z}\|_\infty < \gamma$, where $\|T_{\omega z}\|_\infty$ is the H_∞ norm of $T_{\omega z}$, defined by $\|T_{\omega z}(s)\|_\infty = \sup_{w\in\mathbf{R}} \bar{\sigma}\,(T_{\omega z}(jw))$ (Zhou & Doyle, 1998). The H_∞ performance limit of network (3) is defined as the minimal $\|T_{\omega z}(s)\|_\infty$ of (3) achieved by using controller (2).

Assume hereafter that the communication graph $\mathcal{G}$ is connected. Then, the Laplacian matrix $\mathcal{L}_m$ is positive definite. Denote by $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$ the eigenvalues of $\mathcal{L}_m$.

Theorem 3. *For a given $\gamma > 0$, system (3) is asymptotically stable and $\|T_{\omega z}\|_\infty < \gamma$, if and only if the following N systems are simultaneously asymptotically stable and the H_∞ norms of their transfer function matrices are all less than γ:*

$$\begin{aligned}\dot{\hat{x}}_i &= (A + c\lambda_i B_1K)\hat{x}_i + D_1\hat{\omega}_i,\\ \hat{z}_i &= C\hat{x}_i + D_2\hat{\omega}_i,, \quad i = 1, 2, \ldots, N.\end{aligned} \tag{4}$$

Proof. Let $U \in \mathbf{R}^{N\times N}$ be a unitary matrix such that $U^{-1}\mathcal{L}_m U = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. Let $x = (U \otimes I_n)\hat{x}$, where $\hat{x} = [\hat{x}_1^T, \ldots, \hat{x}_N^T]^T$. Then, system (3) can be rewritten in terms of $\hat{x}$ as

$$\begin{aligned}\dot{\hat{x}} &= (I_N \otimes A + c\Lambda \otimes B_1K)\hat{x} + (U^{-1} \otimes D_1)\omega,\\ z &= (U \otimes C)\hat{x} + (I_N \otimes D_2)\omega.\end{aligned} \tag{5}$$

Further, reformulate the disturbance variable ω and the performance variable z via

$$\omega = (U \otimes I_{m_1})\hat{\omega}, \qquad z = (U \otimes I_{m_2})\hat{z}. \tag{6}$$

Then, substituting (6) into (5) gives

$$\begin{aligned}\dot{\hat{\xi}} &= (I_N \otimes A + c\Lambda \otimes B_1K)\hat{x} + (I_N \otimes D_1)\hat{\omega},\\ \hat{z} &= (I_N \otimes C)\hat{x} + (I_N \otimes D_2)\hat{\omega},\end{aligned} \tag{7}$$

where $\hat{w} = [\hat{\omega}_1, \ldots, \hat{\omega}_N]$, $\hat{z} = [\hat{z}_1, \ldots, \hat{z}_N]$. Note that (7) is composed of the N individual systems in (4). Denote by $T_{\hat{\omega}\hat{z}}$ and $T_{\hat{\omega}_i\hat{z}_i}$ the transfer function matrices of systems (7) and (4), respectively. Then, it follows from (4), (6) and (7) that

$$\begin{aligned}T_{\hat{\omega}\hat{z}} &= \mathrm{diag}(T_{\hat{\omega}_1\hat{z}_1}, \ldots, T_{\hat{\omega}_N\hat{z}_N})\\ &= (U^{-1} \otimes I_{m_1})T_{\omega z}(U \otimes I_{m_2}),\end{aligned} \tag{8}$$

which implies that

$$\|T_{\hat{\omega}\hat{z}}\|_\infty = \max_{i=1,\ldots,N} \|T_{\hat{\omega}_i\hat{z}_i}\|_\infty = \|T_{\omega z}\|_\infty. \tag{9}$$

This completes the proof. □

Remark 1. Theorem 3 converts the distributed H_∞ control problem of the multi-agent network (3) into the H_∞ control problems of a set of independent systems having the same dimensions as a single agent in (1), thereby reducing the computational complexity significantly. The key tools leading to this result rely on the state, the input and the output transformation all together, as used in, e.g., Li et al. (2009) and Massioni and Verhaegen (2009).

Remark 2. For the case where the communication graph $\mathcal{G}$ is a simple graph, i.e., $a_{ii} = 0$ in controller (2), it follows from Lemma 1 that $\lambda_1 = 0$ in (4). The system in (4) corresponding to $\lambda_1 = 0$ is the same as a single agent in (1) with $u_i = 0$. Therefore, the H_∞ performance limit of system (3) in this case is not less than the H_∞ norm of (1) with $u_i = 0$. This implies that at least one state feedback will be needed in controller (2), i.e., graph $\mathcal{G}$ cannot be simple but must have at least one loop in order to reach a better H_∞ performance.

Remark 3. Contrary to the distributed controllers proposed in Borrelli and Keviczky (2008), Fax and Murray (2004), Gupta et al. (2005) and Massioni and Verhaegen (2009), where state feedbacks are required in the controllers to all the agents, controller (2) needs only a subset of agents to know their own states, thereby fully utilizing the favorable effects of relative-state feedbacks. Another unique feature of controller (2) is that by introducing a constant scalar $c > 0$, called the coupling strength, the notions of H_∞ and H_2 performance regions can be brought forward, as detailed in the following sections.

3. H_∞ performance region

Given a controller in the form of (2), the distributed H_∞ problem of network (3) can be recast into analyzing the following system:

$$\begin{aligned}\dot{\zeta} &= (A + \sigma B_1K)\zeta + D_1\omega_i,\\ z_i &= C\zeta + D_2\omega_i,\end{aligned} \tag{10}$$

where $\zeta \in \mathbf{R}^n$ and $\sigma \in \mathbf{R}$, with σ depending on c. The transfer function of system (10) is denoted by $\widetilde{T}_{\omega_i z_i}$. Clearly, the stability and H_∞ performance of system (10) depends on the scalar parameter σ.

The notion of H_∞ performance region is defined as follows.

Definition 1. The region $\mathcal{S}_\gamma$ of the parameter $\sigma \in \mathbf{R}_+$, such that system (10) is asymptotically stable and $\|\widetilde{T}_{\omega_i z_i}\|_\infty < \gamma$, is called the H_∞ performance region with performance index γ of network (3).

The H_∞ performance region can be regarded as an extension of the consensus region introduced in Li et al. (2010) and the synchronization region studied in Pecora and Carroll (1998) and Duan, Chen, and Huang (2008), used to evaluate the performance of a multi-agent network subject to external disturbances. According to Theorem 3, one has the following corollary.

Corollary 4. *Network (3) is asymptotically stable and $\|T_{\omega z}\|_\infty < \gamma$, if and only if $c\lambda_i \in \mathcal{S}_\gamma$, for $i = 1, 2, \ldots, N$.*

For a controller of the form (2), its H_∞ performance region with index γ, if it exists, is an interval or a union of several intervals on the real axis, where the intervals themselves can be either bounded or unbounded. The H_∞ performance region can serve as a measure for the robustness of controller (2) with respect to the communication topology of (3), as illustrated by the several examples given below.

3.1. Examples and analysis

The first example has a bounded H_∞ performance region.

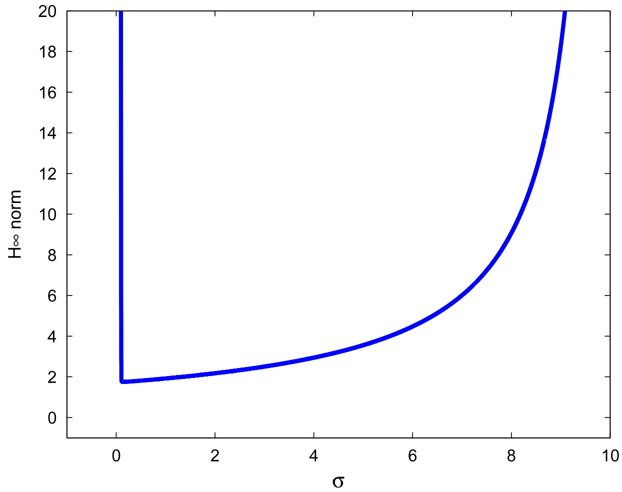

Fig. 1. The H_∞ performance region.

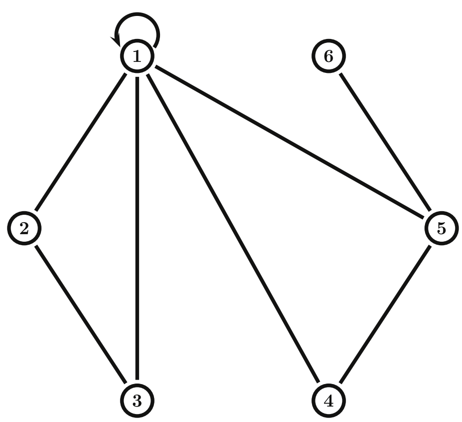

Fig. 2. The communication topology.

Example 1. The agent dynamics and the controller are given by (1) and (2), respectively, with

$$A=\begin{bmatrix}-2 & 1.5\\ -1 & 1\end{bmatrix},\qquad B_1=\begin{bmatrix}1\\ -1\end{bmatrix},\qquad D_1=\begin{bmatrix}1\\ 0.6\end{bmatrix},$$
$$C=\begin{bmatrix}1 & 1.2\end{bmatrix},\qquad D_2=0,\qquad K=\begin{bmatrix}1 & 0.9\end{bmatrix}. \tag{11}$$

The H_∞ performance of (10) with respect to parameter σ is depicted in Fig. 1. It can be observed that $\mathcal{S}_{\gamma>1.751}$, i.e., the H_∞ performance region with index γ larger than the minimal value 1.751, is a bounded interval of σ in $\mathbf{R}$; for example, $\mathcal{S}_{\gamma=3.2}$ is $[0.101635, 4.4668]$.

For illustration, let the communication topology $\mathcal{G}$ be given as in Fig. 2, with Laplacian matrix

$$\mathcal{L}_m=\begin{bmatrix}5 & -1 & -1 & -1 & -1 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ -1 & -1 & 2 & 0 & 0 & 0\\ -1 & 0 & 0 & 2 & -1 & 0\\ -1 & 0 & 0 & -1 & 3 & -1\\ 0 & 0 & 0 & 0 & -1 & 1\end{bmatrix}.$$

The minimal and maximal eigenvalues of $\mathcal{L}_m$ are 0.1355 and 5.8928, respectively. Thus, controller (2) solves the suboptimal H_∞ problem with $\gamma=3.2$ for the graph given in Fig. 2 if and only if c (converted from σ) lies within the set $[0.7501, 0.7580]$.

Let us see how modifications of the communication topology affect the H_∞ performance by considering the following simple cases.

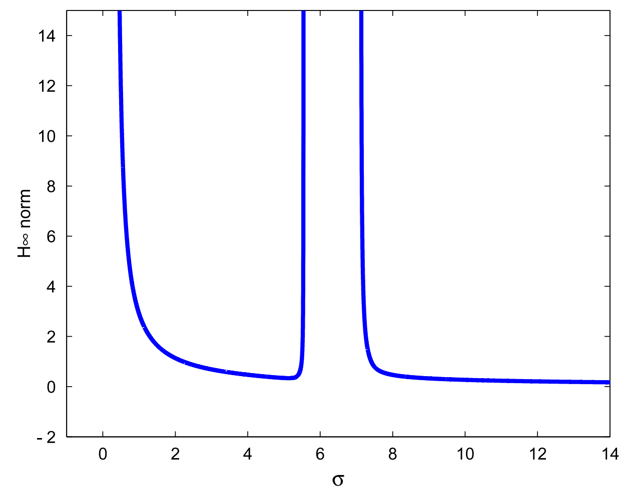

Fig. 3. The disconnected H_∞ performance region.

- An edge between node 1 and node 6 is added, i.e., more information exchanges exist inside the network. The minimal and maximal eigenvalues of the resulting Laplacian matrix are 0.1459 and 6.8541, respectively. Thus, it can be verified that controller (2) fails to solve the suboptimal H_∞ problem with $\gamma=3.2$ for any $c\in\mathbf{R}_+$.
- The edge between node 4 and node 5 is removed. The minimal and maximal eigenvalues of the resulting Laplacian matrix become 0.1336 and 5.8857, respectively. The controller (2) in this case also fails to solve the suboptimal H_∞ problem with $\gamma=3.2$.
- The loop around node 1 in Fig. 2 is moved to node 6, i.e., the self-feedback part in (2) acts on agent 6 rather than on agent 1. The minimal and maximal eigenvalues of the resulting Laplacian matrix are 0.0788 and 5.1186, respectively. Once again, the controller fails to solve the suboptimal H_∞ problem with $\gamma=3.2$.

These sample cases imply that the distributed controller (2), if not well designed, can be quite fragile to variations of the network's communication topology. In other words, it is desirable for the H_∞ performance region to be large enough in order to ensure that the controller maintain a desired robustness margin with respect to the communication topology.

The second example has a disconnected H_∞ performance region.

Example 2. The agent dynamics and the controller are given by (1) and (2), respectively, with

$$A=\begin{bmatrix}-0.32 & -20.6\\ 4.6 & 0.4\end{bmatrix},\qquad B_1=\begin{bmatrix}1.5 & 0.3\\ 0 & 1\end{bmatrix},\qquad D_1=\begin{bmatrix}1\\ 0\end{bmatrix},$$
$$C=\begin{bmatrix}0 & 1\end{bmatrix},\qquad D_2=0,\qquad K=\begin{bmatrix}0 & 2\\ -0.8 & 0\end{bmatrix}.$$

The H_∞ performance region of (10) with respect to parameter σ is depicted in Fig. 3, which is composed of two disjoint subregions, of which one is bounded and the other is unbounded. It can be verified that $\mathcal{S}_{\gamma=3.2}=[0.98, 5.5267]\cup[7.2116,\infty)$. For the communication topology given in Fig. 2, the controller (2) solves the suboptimal H_∞ problem with $\gamma=3.2$, if and only if $c\geq 53.2221$ (after converting σ to c).

3.2. H_∞ performance limit and synthesis

It was shown in the last subsection that the distributed controller should have a large enough H_∞ performance region to be robust with respect to the communication topology. One convenient and desirable choice is to design the controller with an unbounded H_∞ performance region.

Lemma 5 (*Bounded Real Lemma Zhou & Doyle, 1998*)**.** *Let $\gamma > 0$ and $G(s) = C(sI - A)^{-1}B + D$. Then, the following two statements are equivalent:*

(1) *A is stable and $\|G(s)\|_\infty < \gamma$.*
(2) *$\bar{\sigma}(D) < \gamma$ and there exists an $X > 0$ such that*

$$AX + XA^T + XC^TCX + (B + XC^TD) \times (\gamma^2 - D^TD)^{-1}(B^T + D^TCX) < 0.$$

In the following, a necessary and sufficient condition is derived for the existence of a controller (2) having an unbounded H_∞ performance region.

Theorem 6. *For a given $\gamma > 0$, there exists a controller (2) having an unbounded H_∞ performance region if and only if there exist a matrix $P > 0$ and a scalar $\tau > 0$ such that*

$$\begin{bmatrix} AP + PA^T - \tau B_1B_1^T & D_1 & PC^T \\ D_1^T & -\gamma^2 I & D_2^T \\ CP & D_2 & -I \end{bmatrix} < 0. \tag{12}$$

Moreover, the unbounded H_∞ performance region $\mathcal{S}_\gamma$ contains $[\tau, \infty)$.

Proof. For simplicity, only the special case with $D_2 = 0$ is discussed here. The proof for the case of $D_2 \neq 0$ is quite similar, just notationally more involved.

(Necessity) According to Definition 1, if network (3) has an unbounded H_∞ performance region, then $A + \sigma B_1 K$ is Hurwitz and $\|C(sI - A - \sigma B_1K)^{-1}D_1\|_\infty < \gamma$ for some matrix K and scalar σ. Since K is to be designed, without loss of generality, choose $\sigma = 1$. By Lemma 5, there exists a matrix K such that $A + B_1K$ is Hurwitz, and $\|C(sI - A - B_1K)^{-1}D_1\|_\infty < \gamma$, if and only if there exists a matrix $P > 0$ such that

$$(A + B_1K)P + P(A + B_1K)^T + \frac{1}{\gamma^2}D_1D_1^T + PC^TCP < 0.$$

Let $Y = KP$. Then, the above inequality becomes

$$AP + PA^T + B_1Y + Y^TB_1^T + \frac{1}{\gamma^2}D_1D_1^T + PC^TCP < 0.$$

By Finsler's Lemma (Iwasaki & Skelton, 1994), there exists a matrix Y satisfying the above inequality if and only if there exists a scalar $\tau > 0$ such that

$$AP + PA^T - \tau B_1B_1^T + \frac{1}{\gamma^2}D_1D_1^T + PC^TCP < 0, \tag{13}$$

which, in virtue of the Schur Complement Lemma (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994), is equivalent to

$$\begin{bmatrix} AP + PA^T - \tau B_1B_1^T & D_1 & PC^T \\ D_1^T & -\gamma^2 I & 0 \\ CP & 0 & -I \end{bmatrix} < 0. \tag{14}$$

(Sufficiency) If (14) holds for some matrix $P > 0$ and scalar $\tau > 0$, then (13) holds also. Take $K = -\frac{1}{2}B_1^TP^{-1}$. Then, for $c\lambda_i \geq \tau$, $i = 1, 2, \ldots, N$, it follows from (13) that

$$\begin{aligned}(A + c\lambda_iB_1K)P + P(A + c\lambda_iB_1K)^T + \frac{1}{\gamma^2}D_1D_1^T + PC^TCP \\ = AP + PA^T - c\lambda_iB_1B_1^T + \frac{1}{\gamma^2}D_1D_1^T + PC^TCP < 0,\end{aligned} \tag{15}$$

implying that $\|C(sI - A - c\lambda_iB_1K)^{-1}D_1\|_\infty < \gamma$, $i = 1, 2, \ldots, N$, i.e., controller (2) with K given as above has an unbounded H_∞ performance region containing $[\tau, \infty)$. □

The exact H_∞ performance limit of network (3) under controller (2) is now obtained as a consequence.

Corollary 7. *The H_∞ performance limit $\gamma_{\min}$ of network (3) under controller (2) is given by the following optimization problem:*

$$\begin{aligned}&\text{minimize } \gamma \\ &\text{subject to LMI (12)}, \quad \text{with } P > 0,\ \tau > 0,\ \gamma > 0.\end{aligned} \tag{16}$$

Remark 4. Note that the H_∞ performance limit $\gamma_{\min}$ of network (3), consisting of N agents in (1) under controller (2), is actually equal to the minimal H_∞ norm of a single agent (1) by using a state feedback controller of the form $u_i = Fx_i$, independent of the communication topology $\mathcal{G}$ as long as it is connected. To the best of the authors' knowledge, this is the first time that the exact H_∞ performance limit of distributed control for linear multi-agent systems is derived. Comparing to related works Li et al. (2009) and Massioni and Verhaegen (2009), the conditions given there are relatively conservative. It should be mentioned that the performance output z_i is assumed here not to directly depend upon u_i, while the designs in Li et al. (2009) and Massioni and Verhaegen (2009) allow putting a penalty on the control law and are applicable even to output feedback controllers. Moreover, the minimum $\gamma_{\min}$ achieved by solving LMI (16) generally corresponds to a high-gain controller (2), as depicted in the example below. Choosing a γ a little bigger than $\gamma_{\min}$ would help keep the gain smaller.

A procedure for the H_∞ controller synthesis is now presented.

Algorithm 1. For any $\gamma \geq \gamma_{\min}$, where $\gamma_{\min}$ is given by (16), the controller (2) solving the distributed H_∞ control problem can be constructed as follows:

(1) Solve LMI (12) for a feasible solution: $P > 0$ and $\tau > 0$.
(2) Choose the feedback gain matrix $K = -\frac{1}{2}B_1^TP^{-1}$.
(3) Select the coupling strength c not less than the threshold value $c_{th} = \frac{\tau}{\min_{i=1,\ldots,N} \lambda_i}$, where λ_i, $i = 1, 2, \ldots, N$, are the eigenvalues of $\mathcal{L}_m$.

Remark 5. The above design procedure for constructing a distributed H_∞ controller has a favorable decoupling feature. Specifically, steps (1) and (2) design the feedback gain matrix K of controller (2) to yield an unbounded H_∞ performance region, dealing only with the agent dynamics, while step (3) adjusts the coupling strength c such that $c\lambda_i$, $i = 1, 2, \ldots, N$, lie in this region. This feature is very desirable for the case where the agent number N is large, for which the eigenvalues of the corresponding Laplacian matrix $\mathcal{L}_m$ are hard to determine or even troublesome to estimate. Here, one only needs to choose the coupling strength to be large enough.

Example 3. The agent dynamics are given by

$$A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0.8 \end{bmatrix}, \quad D_2 = 0.$$

Solving LMI (12) with $\gamma = 1$ by using toolboxes Yalmip (Löfberg, 2004) and SeDuMi (Sturm, 1999) give a feasible solution: $P = \begin{bmatrix} 1.239 & 0.1371 \\ 0.1371 & 0.2015 \end{bmatrix}$ and $\tau = 1.5325$. Thus, the feedback gain matrix is chosen as $K = \begin{bmatrix} -0.7332 & 2.98 \end{bmatrix}$. By Algorithm 1, controller (2) with this matrix K has an unbounded H_∞ performance region with index $\gamma = 1$ in the form of $[1.5325, \infty)$. This can also be verified in another way by depicting the H_∞ norm of system (10) with respect to scalar σ in Fig. 4, from which it can be observed that $\mathcal{S}_{\gamma=1}$

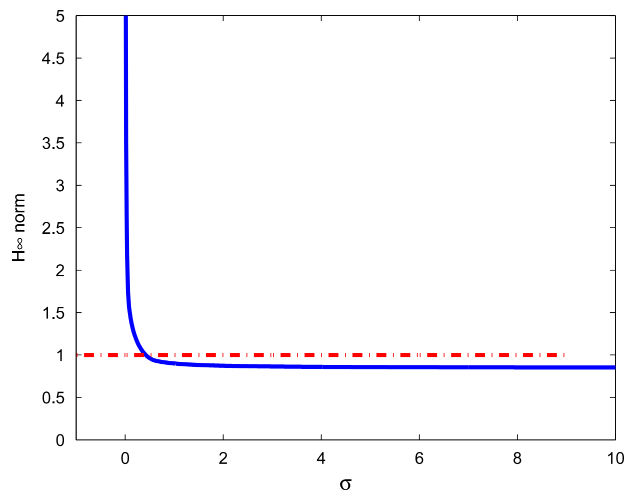

Fig. 4. The unbounded H_∞ performance region.

contains the region $[1.5325, \infty)$. For the graph in Fig. 2, controller (2) with K chosen above solves the suboptimal H_∞ control problem with $\gamma = 1$, if the coupling strength c is not less than the threshold value $c_{th} = 11.31$.

By solving the optimization problem (16), the H_∞ performance limit of network (3) under controller (2) can be further obtained, as $\gamma_{\min} = 0.0546$. The corresponding optimal feedback gain matrix of (2) is obtained as $K = \begin{bmatrix} 2.8018 & 2.8154 \end{bmatrix}$, and the scalar τ in (12) is $\tau = 2.2202 \times 10^6$. For the graph in Fig. 2, the threshold c_{th} corresponding to $\gamma_{\min}$ is $c_{th} = 1.6385 \times 10^7$ in this numerical example.

4. H_2 performance region

In this section, the H_2 performance of network (3) is discussed. For a stable transfer function $F(s)$, its H_2 norm is defined as $\|F\|_2 = \{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}[F^*(j\omega)F(j\omega)]\mathrm{d}\omega\}^{\frac{1}{2}}$ (Zhou & Doyle, 1998). In this case, matrix D_2 in (1) needs to be zero in order to guarantee the existence of the H_2 norm of network (3).

Similar to the H_∞ case, the suboptimal H_2 control problem is stated as follows: for a given $\tilde{\gamma} > 0$, find a controller (2) such that (i) network (3) is asymptotically stable; (ii) $\|T_{\omega z}\|_2 < \tilde{\gamma}$. The H_2 performance limit of network (3) is the minimal $\|T_{\omega z}\|_2$ of (3) under controller (2).

Theorem 8. *For a given $\tilde{\gamma} > 0$, network (3) is asymptotically stable with $\|T_{\omega z}\|_2 < \tilde{\gamma}$, if and only if the N systems in (4) with $D_2 = 0$ are simultaneously asymptotically stable and $\sqrt{\sum_{i=1}^{N}\|T_{\hat{\omega}_i\hat{z}_i}\|_2^2} < \tilde{\gamma}$, where $T_{\hat{\omega}_i\hat{z}_i}$, $i = 1, \ldots, N$, are the transfer function matrices of the systems in (4).*

Proof. It follows readily from (8) and the definition of the H_2 norm. □

The H_2 performance region is defined as follows.

Definition 2. The region $\widetilde{\mathcal{S}}_{\tilde{\gamma}}$ of the parameter $\sigma \in \mathbf{R}_+$, such that system (10) with $D_2 = 0$ is asymptotically stable, with $\|\widetilde{T}_{\omega_i z_i}\|_2 < \frac{\tilde{\gamma}}{\sqrt{N}}$, is called the H_2 performance region with performance index $\tilde{\gamma}$ of network (3).

According to Theorem 8, one has the following corollary.

Corollary 9. *Network (3) with $D_2 = 0$ is asymptotically stable and $\|T_{\omega z}\|_2 < \tilde{\gamma}$, if $c\lambda_i \in \widetilde{\mathcal{S}}_{\tilde{\gamma}}$, for $i = 1, 2, \ldots, N$.*

Remark 6. Contrary to the H_∞ case, the H_2 performance region is related to the number of agents in the network and using H_2 performance region to characterize the H_2 performance of network (3) involves certain conservatism. This is essentially due to the inherent difference between the H_2 and H_∞ norms, and due to the fact that, in the H_2 case, the N systems in (4) are coupled with each other, which thereby is more difficult to analyze.

The H_2 performance region analysis can be discussed similarly to Section 3, therefore is omitted here for brevity. The synthesis issue is somewhat different, hence is further discussed below.

Lemma 10 (Zhou & Doyle, 1998). *Let $\tilde{\gamma} > 0$ and $G(s) = C(sI - A)^{-1}B$. Then, the following two statements are equivalent.*

(1) *A is stable and $\|G(s)\|_2 < \tilde{\gamma}$.*
(2) *there exists an $X > 0$ such that*

$$AX + XA^T + BB^T < 0, \qquad \mathrm{tr}(CXC^T) < \tilde{\gamma}^2.$$

Theorem 11. *For a given $\tilde{\gamma} > 0$, there exists a distributed controller (2) having an unbounded H_2 performance region if and only if there exist a matrix $Q > 0$ and a scalar $\tilde{\tau}$ such that*

$$\begin{aligned} &AQ + QA^T - \tilde{\tau}B_1B_1^T + D_1D_1^T < 0, \\ &\mathrm{tr}(CQC^T) < \frac{\tilde{\gamma}^2}{N}. \end{aligned} \tag{17}$$

Moreover, the unbounded H_2 performance region $\widetilde{\mathcal{S}}_{\tilde{\gamma}}$ contains $[\tilde{\tau}, \infty)$.

Proof. (Necessity) Similar to the proof of Theorem 6, if network (3) has an unbounded H_2 performance region, then there exists a matrix K such that $A + B_1K$ is Hurwitz and $\|C(sI - A - B_1K)^{-1}D_1\|_2 < \frac{\tilde{\gamma}}{\sqrt{N}}$, which is equivalent to that there exists a matrix $Q > 0$ such that

$$\begin{aligned} &(A + B_1K)Q + Q(A + B_1K)^T + D_1D_1^T < 0, \\ &\mathrm{tr}(CQC^T) < \frac{\tilde{\gamma}^2}{N}. \end{aligned}$$

Let $V = KQ$. Then, the above inequality becomes

$$\begin{aligned} &AQ + QA^T + B_1V + V^TB_1^T + D_1D_1^T < 0, \\ &\mathrm{tr}(CQC^T) < \frac{\tilde{\gamma}^2}{N}, \end{aligned}$$

which, by Finsler's Lemma, is equivalent to that there exist a matrix $Q > 0$ and a scalar $\tilde{\tau} > 0$ such that (17) holds.

(Sufficiency) Take $K = -\frac{1}{2}B_1^TQ^{-1}$. For $c\lambda_i \geq \tilde{\tau}$, $i = 1, 2, \ldots, N$, one has

$$\begin{aligned} &(A + c\lambda_iB_1K)Q + Q(A + c\lambda_iB_1K)^T + D_1D_1^T \\ &\quad = AQ + QA^T - c\lambda_iB_1B_1^T + D_1D_1^T < 0, \\ &\mathrm{tr}(CQC^T) < \frac{\tilde{\gamma}^2}{N}, \end{aligned}$$

which together imply that $\|C(sI - A - c\lambda_iB_1K)^{-1}D_1\|_2 < \frac{\tilde{\gamma}}{\sqrt{N}}$, $i = 1, \ldots, N$, i.e., controller (2) with K chosen as above has an unbounded H_2 performance region containing $[\tilde{\tau}, \infty)$. □

Corollary 12. *The H_2 performance limit $\tilde{\gamma}_{\min}$ of network (3) under controller (2) is given by the optimization problem:*

$$\begin{aligned} &\textit{minimize } \tilde{\gamma} \\ &\text{subject to LMI (17)}, \quad \textit{with } Q > 0,\ \tilde{\tau} > 0,\ \tilde{\gamma} > 0. \end{aligned} \tag{18}$$

Proof. Solving the optimization problem (18) gives solutions $\tilde{\gamma}_{\min}$ and $\tilde{\tau}_{\min}$. Select c such that $c\lambda_i \geq \tilde{\tau}_{\min}$, for $i = 1, 2, \ldots, N$. Then, controller (2) with K given as in the proof of Theorem 11 yields $\|T_{\hat{\omega}_i\hat{z}_i}\|_2 = \frac{\tilde{\gamma}_{\min}}{\sqrt{N}}$, for $i = 1, 2, \dots, N$, which by Theorem 8 imply that $\tilde{\gamma}_{\min}$ is the H_2 performance limit of network (3). □

Remark 7. The H_2 performance limit of network (3) under controller (2) is related to two factors: the minimal H_2 norm of a single agent (1) by using the state feedback controller $u_i = Fx_i$ and the number of agents in the network. Contrary to the H_∞ case, the H_2 performance limit of network (3) scales with the size of the network.

Algorithm 2. For any $\tilde{\gamma} \geq \tilde{\gamma}_{\min}$, where $\tilde{\gamma}_{\min}$ is given by (16), the controller (2) solving the distributed H_2 control problem can be constructed as follows:

(1) Solve LMI (17) for obtaining a solution: $Q > 0$ and $\tilde{\tau} > 0$.
(2) Choose the feedback gain matrix $K = -\frac{1}{2}B_1^T Q^{-1}$.
(3) Select the coupling strength $c > \tilde{c}_{th}$, with $\tilde{c}_{th} = \frac{\tilde{\tau}}{\min\limits_{i=1,\dots,N} \lambda_i}$, where $\lambda_i, i = 1, 2, \dots, N$, are the eigenvalues of $\mathcal{L}_m$.

5. Conclusions

This paper has studied the distributed H_2 and H_∞ control problems of multi-agent systems with linear or linearized dynamics. The distributed controllers have been designed for the two problems, respectively, based on the relative states of neighboring agents and a subset of absolute states of the agents in the network. The novel notions of H_∞ and H_2 performance regions have been introduced and analyzed. A necessary and sufficient condition for the existence of a controller having an unbounded H_∞ performance region has been derived, which ensure better robustness of the controller and the controlled network. A simple procedure for suboptimal H_∞ and H_2 controller synthesis has been also presented. The exact H_∞ performance limit of the network under the distributed controller is equal to the minimal H_∞ norm of an individual agent achieved by using the state feedback controller. It has been shown that, contrarily to the H_∞ case, the H_2 performance limit scales with the size of the network.

Acknowledgements

The authors would like to thank the Associate Editor and all the reviewers for their constructive suggestions. This work was supported in part by the National Natural Science Foundation of China under Grants 60974078, 10832006 and 10872030, and by China Postdoctoral Science Foundation.

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