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Continuous-time model identification of fractional order models with time delays

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Abstract: This paper deals with the continuous-time model identification (CMI) of fractional order systems with time delays. In this paper, a new linear filter is introduced for simultaneous estimation of all model parameters for commensurate fractional order systems with time delays (CFOTDS) based on step response data. The proposed method simultaneously estimates the time delay along with other model parameters in an iterative manner by solving simple linear regression equations. For the case when the fractional order is unknown, we also propose a nested loop optimization method where the time delay along with other model parameters are estimated iteratively in the inner loop and the fractional order is estimated in the non-linear outer loop. The applicability of the developed procedure is demonstrated on two fractal systems by doing Monte Carlo simulation analysis in the presence of white noise.

Keywords: fractional order systems; continuous-time model identification; instrumental variable; step response; time delays; nested loop optimization.

1. INTRODUCTION

Fractional calculus is a generalization of the traditional integer order integral and differential calculus to non-integer orders. With the growing power of computers, fractional calculus now has become an increasingly interesting topic of research in the scientific and industrial communities. Before the 19th century, the theory of fractional calculus developed mainly as a pure theoretical field of mathematics useful only for mathematicians. A significant amount of discussions aimed at this subject has been presented by Oldham and Spanier (1974) and Podlubny (1999). However, recently it has been observed that many real-world physical systems are well characterized by fractional-order differential equations rather than using classical integer order models. In particular, materials having long memory and hereditary effects (Bagley and Torvik (1984)) and dynamical processes such as mass diffusion and heat conduction (Jenson and Jeffreys (1974)) in fractal porous media can be more adequately modeled by fractional-order models rather than integer-order models. Some of the other examples of fractal systems include transmission lines, electrochemical processes, dielectric polarization and viscoelastic materials. The special issue of signal processing (Ortigueira and Tenreiro (2006)) discusses many of the applications of fractional calculus in detail.

System identification has become the standard tool for modeling unknown systems. However, identifying a given system from data becomes more difficult when the physical systems are characterized by fractional-order differential equations instead of classical integer order models. Thus, fractional models, using fractional differentiation, have been developed. Time-domain system identification using fractional differentiation models was initiated by Lay (Lay (1998)) and Cois (Cois (2002)), in their PhD thesis work in the late 1990s. The two model identification approaches developed were: Equation-error-based and output-error-based approaches, both of which are very well studied in the literature. As in the case of continuous model identification for integer order models, fractional differentiation of the noisy signals also amplifies noise. Hence, a linear transformation using low-pass filter can be applied to the model equation. As for the integer case, there are many filters proposed for FO models such as fractional integral filter, Poisson's state variable (SVFs) filters (Cois et al. (2001)), and Refined Instrumental Variable for Continuous systems (RIVC) (Malti et al. (2008*a*)). Also, identification methods based on orthogonal basis functions (fractional Laguerre and Kautz basis functions) have been proposed (Aoun et al. (2007)). The recent paper by Malti et al. (2008*b*) discusses briefly all these advances in time-domain system identification using fractional models.

However none of these studies discuss methods for identification of fractional-order systems with time delays. For example, due to the actuator limitations in some systems such as motion control, it is reported in Manabe (2003) that the system can be well modeled with a fractional-order open-loop transfer function with time delay. In this paper, we describe a scheme for continuous time identification of commensurate fractional order models with time delays based on step response. In this scheme the delay is estimated simultaneously with other model parameters. The formulation as proposed by Ahmed et al. (2007) for integer order continuous-time systems is extended to the identification of fractional models. To the best knowledge of the authors no formulation for estimating all model parameters including delays has been proposed for fractal models. The formulation is based on the low pass filtering operation where the filter is chosen as the combination of RIVC and a linear integral filter, to decouple the delay term from other parameters. The proposed method estimates the time delay along with constant model parameters in an iterative manner by solving simple linear regression equations. We also propose a nested loop optimization method where the time delay along with constant model parameters are estimated iteratively in the inner loop and the fractional order is estimated in the non-linear outer loop.

This paper is organized as follows. Section 2 presents a brief mathematical background of fractional calculus with an introduction to fractional order models. The continuous time model identification algorithm for CFOTDS for step input signals is presented in Section 3. To study the efficacy of the proposed strategy developed in the Section 3, two example of fractal models in the presence of noise are outlined in section 4 to demonstrate its applicability followed by concluding remarks in Section 5.

2. MATHEMATICAL BACKGROUND

2.1 Definitions and FO models

Fractional calculus is a generalization of integration and differentiation to non-integer orders. The two most popular definitions used for the general fractional differintegral are the Grünwald-Letnikov (GL) discrete form of the definition and the Riemann-Liouville (RL) definition (Oldham and Spanier (1974)). The GL definition for a function $f(t)$ is given as

$$\mathbf{D}^{\lambda} f(t) = \lim_{n\to\infty} \frac{1}{h^{\lambda}} \sum_{i=0}^{\infty} [(-1)^i \begin{pmatrix} \lambda \\ i \end{pmatrix} f(t-ih) \tag{1}$$

where

$$\begin{pmatrix} \lambda \\ i \end{pmatrix} = \frac{\mathbf{\Gamma}(\lambda+1)}{\mathbf{\Gamma}(i+1)\mathbf{\Gamma}(\lambda-i+1)} \tag{2}$$

and the operator $\mathbf{D}^{\lambda}$ defines fractional differentiation or integration depending on the sign of λ, $\mathbf{\Gamma}(.)$ being the well known Euler's Gamma function and h is the finite sampling interval. This definition is particularly useful for digital implementation of fractional order controllers. The RL definition is given as

$$\mathbf{D}^{\lambda} f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\lambda)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\lambda+1-m}} d\tau\right] \tag{3}$$

where m is an integer such that $(m-1 < \lambda < m)$ and $t > 0 \ \forall \ \lambda \in \mathbb{R}_+$

For convenience, the Laplace domain notation is usually used to describe fractional differ-integral operation. When the initializations are assumed to be zero,

$$\mathbf{L}\{D^{\lambda} f(t)\} = s^{\lambda} F(s) \qquad (\lambda \in \mathbb{R}) \tag{4}$$

The generic single-input single-output (SISO) fractional order system representation in the Laplace domain is given as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^{\beta_0} + b_1 s^{\beta_1} + ... + b_m s^{\beta_m}}{1 + a_1 s^{\alpha_1} + ... + a_n s^{\alpha_n}} \tag{5}$$

where $b_0, b_1, ..., b_m$ and $a_1, a_2, ..., a_n$ are constant model parameters or model coefficients, while $\beta_0 < \beta_1 < ..... < \beta_m$ and $\alpha_1 < \alpha_2 < ..... < \alpha_n$ are the fractional powers or fractional orders (real numbers).

The transfer function given by equation (5) can be classified as either a commensurate transfer function or a non-commensurate transfer function. A transfer function $G(s)$ is commensurable of order γ if and only if it can be written as $G(s) = F(s^{\gamma})$, where $F = T/R$ is a rational function, with T and R as two coprime polynomials. Assuming that $G(s)$ is commensurable of order γ, then it can be written as

$$G(s) = \frac{\sum_{j=0}^{m} b_j s^{j\gamma}}{1 + \sum_{i=1}^{n} a_i s^{i\gamma}} \tag{6}$$

where we substitute $\beta_j = j\gamma$ and $\alpha_i = i\gamma$. The transfer function (5) is called non-commensurate when β_j, α_i can take any arbitrary values. On the other hand, commensurate transfer function models represent more generic class of polynomial type transfer functions where $\gamma = 1$ gives standard integer order transfer function models. A commensurate transfer function of order γ for fractional-order time delay system is given as

$$G(s) = \frac{\sum_{j=0}^{m} b_j s^{j\gamma}}{1 + \sum_{i=1}^{n} a_i s^{i\gamma}} \mathrm{e}^{-Ls} \tag{7}$$

where L is the time delay. In this work we will be working only with commensurate transfer function models with delays as given by (7).

2.2 Stability condition

Stability condition for a class of transfer function of the form (6) has been established by Matignon (1998). The theorem is as follows:

Stability Theorem *A commensurable $\gamma-$order transfer function $G(s) = F(s^{\gamma}) = T(s^{\gamma})/R(s^{\gamma})$, where $T(.)$ and $R(.)$ are two coprime polynomials, is BIBO stable if and only if*

$$0 < \gamma < 2$$

and for $\sigma \in \mathcal{C}$ such that $R(\sigma) = 0$

$$|\arg(\sigma)| > \gamma\frac{\pi}{2}$$

2.3 Integer order approximation

For digital implementation of the fractional order operator, the key step is numerical evaluation or discretization of this operator. Power series expansion and continued fraction expansion (CFE) of the Euler's, Tustin and Al-Alaoui operators give different discrete approximations of the fractional operator. The power series expansion of Euler's operator gives numerical approximation of GL definition (1). The GL definition is the most widely used and implemented discrete approximation for this operator.

The details for the discretization schemes can be found in Vinagre et al. (2002) and Chen and Moore (2002). However, sometimes frequency domain fitting in the continuous time domain of this fractional order operator is done first and then discretization of this transfer function is done to get the discrete approximation. One of the good continuous approximation for this fractional order operator is the Oustaloup continuous approximation (Oustaloup (1995)) where it makes use of a recursive distribution of poles and zeroes. We will be using the numerical approximation of (1) for simulation of fractional order systems.

3. IDENTIFICATION OF FRACTAL SYSTEMS WITH DELAYS

3.1 Identification formulation

The transfer function for CFOTDS of commensurate order α is given as

$$G(s) = \frac{\sum_{j=0}^{m} b_j s^{j\alpha}}{1 + \sum_{j=1}^{n} a_j s^{j\alpha}} \, \mathrm{e}^{-Ls} \tag{8}$$

In rational transfer functions (integer order models) α = 1 and only the coefficients a_j, b_j and L are estimated. However, here we are interested in estimating α as well.

For the present case, initial conditions are assumed zero. The above model in the vector form can be written as

$$\mathbf{a}_n \mathbf{s}^{n\alpha} Y(s) = \mathbf{b}_m \mathbf{s}^{m\alpha} U(s) e^{-Ls} + e(s) \tag{9}$$

where

$$\mathbf{a}_n = [a_n \; a_{n-1} \; .... \; a_1 \; 1] \quad \in \; R^{1\times(n+1)} \tag{10}$$

$$\mathbf{b}_m = [b_m \; b_{m-1} \; .... \; b_1 \; b_0] \quad \in \; R^{1\times(m+1)} \tag{11}$$

$$\mathbf{s}^{n\alpha} = [s^{n\alpha} \; s^{(n-1)\alpha} \; .... \; s^{\alpha} \; s^0]^T \quad \in \; R^{(n+1)\times 1} \tag{12}$$

and $Y(s), U(s)$ and $e(s)$ are the Laplace transforms of output $y(t)$, input $u(t)$ and $e(t)$ respectively. Here $e(t)$ accounts for the noise.

Next, we will devise a linear filter method for the estimation of the parameters. To obtain explicit appearance of the delay term in the estimation equation and get it as an element in the parameter vector, we introduce a linear filter method with structure of the filter as a combination of RIVC and a linear integral filter. This low pass filter not only serves the purpose of removing noise amplification but it also decouples the delay term from the other parameters. The filter transfer function is represented as

$$F(s^{\alpha}) = \frac{1}{sA(s^{\alpha})} \tag{13}$$

where $A(s^{\alpha})$ is the denominator of the above model.

Now applying filtering operation on both sides of equation (9) yields

$$\mathbf{a}_n \mathbf{s}^{n\alpha} F(s^{\alpha}) Y(s) = \mathbf{b}_m \mathbf{s}^{m\alpha} F(s^{\alpha}) U(s) e^{-Ls} + F(s) e(s) \tag{14}$$

or

$$\mathbf{a}_n \mathbf{s}^{n\alpha} \frac{1}{sA(s^{\alpha})} Y(s) = \mathbf{b}_m \mathbf{s}^{m\alpha} \frac{1}{sA(s^{\alpha})} U(s) e^{-Ls} + \varsigma(s) \tag{15}$$

where $\varsigma(s) = F(s)e(s)$.

Here F(s) can be factored as

$$\frac{1}{sA(s^{\alpha})} = \frac{C(s^{\alpha})}{sA(s^{\alpha})} + \frac{1}{s} \tag{16}$$

where

$$C(s^{\alpha}) = -(a_n s^{n\alpha} + a_{n-1} s^{(n-1)\alpha} .....a_1 s^{\alpha}) \tag{17}$$

Also, $\mathbf{a_n}$ and $\mathbf{b_m}$ can be factored as

$$\mathbf{a}_n \mathbf{s}^{n\alpha} = (\bar{\mathbf{a}}_n \mathbf{s}^{(n-1)\alpha} s^{\alpha} + 1) \tag{18}$$

and

$$\mathbf{b}_m \mathbf{s}^{m\alpha} = (\bar{\mathbf{b}}_m \mathbf{s}^{(m-1)\alpha} s^{\alpha} + b_0) \tag{19}$$

where $\bar{\mathbf{a}}_n$ *and* $\bar{\mathbf{b}}_m$ are the $\mathbf{a}_n$ and $\mathbf{b}_m$ vectors respectively with the last element removed. Now defining the filtered output variables as

$$Y_f(s) = \frac{Y(s)}{sA(s^{\alpha})} \;\; and \;\; Y_{fD}(s) = \frac{s^{\alpha} Y(s)}{sA(s^{\alpha})} \tag{20}$$

and similarly for U(s).

Thus (15) becomes

$$Y_f(s) = -\bar{\mathbf{a}}_n \mathbf{s}^{(n-1)\alpha} Y_{fD}(s) + \bar{\mathbf{b}}_m \mathbf{s}^{(m-1)\alpha} U_{fD}(s) e^{-Ls} + b_0 \left(\frac{C(s^{\alpha})}{sA(s^{\alpha})} + \frac{1}{s} \right) U(s) e^{-Ls} + \varsigma(s) \tag{21}$$

and for step input of step size h,

$$U(s) = \frac{h}{s} \tag{22}$$

then

$$Y_f(s) = -\bar{\mathbf{a}}_n \mathbf{s}^{(n-1)\alpha} Y_{fD}(s) + \bar{\mathbf{b}}_m \mathbf{s}^{(m-1)\alpha} U_{fD}(s) e^{-Ls} + b_0 \left(\frac{C(s^{\alpha})}{sA(s^{\alpha})} \right) \frac{h}{s} e^{-Ls} + \; b_0 \frac{1}{s} \frac{h}{s} e^{-Ls} + \varsigma(s) \tag{23}$$

or

$$Y_f(s) = -\bar{\mathbf{a}}_n \mathbf{s}^{(n-1)\alpha} Y_{fD}(s) + \bar{\mathbf{b}}_m \mathbf{s}^{(m-1)\alpha} U_{fD}(s) e^{-Ls} + b_0 C(s^{\alpha}) F(s^{\alpha}) \frac{h}{s} e^{-Ls} + b_0 \frac{1}{s} \frac{h}{s} e^{-Ls} + \varsigma(s) \tag{24}$$

Before taking the Laplace inverse of (24) on both sides, we define the inverse Laplace transforms for various terms as

$$\pounds^{-1}(Y_f(s)) = Y_f(t) \tag{25}$$

$$\pounds^{-1}(Y_{fD}(s)) = Y_{fD}(t) \tag{26}$$

$$\pounds^{-1}(U_{fD}(s)) = U_{fD}(t) \tag{27}$$

$$\pounds^{-1}(F(s^{\alpha})) = F(t) \tag{28}$$

$$\pounds^{-1}(\varsigma(s)) = \varsigma(t) \tag{29}$$

$$\pounds^{-1}\left(\mathbf{s}^{(n-1)\alpha} Y_{fD}(s)\right) = Y_{fD}^{(n-1)\alpha}(t) \tag{30}$$

$$\pounds^{-1}(F(s^{\alpha}) C(s^{\alpha})) = F_C(t) \tag{31}$$

$$\pounds^{-1}(F(s^{\alpha}) C(s^{\alpha}) \frac{h}{s} e^{-Ls}) = h F_C^{I}(t - L) \tag{32}$$

$$\pounds^{-1}\left(\mathbf{s}^{(m-1)\alpha} U_{fD}(s) e^{-Ls}\right) = U_{fD}^{(m-1)\alpha}(t - L) \tag{33}$$

$$\pounds^{-1}\left(\frac{U(s) e^{-Ls}}{s}\right) = \pounds^{-1}\left(\frac{h}{s}\frac{1}{s} e^{-Ls}\right) \tag{34}$$

$$= h\,(t - L) \quad for \quad t > L \tag{35}$$

Now taking inverse Laplace transform of (24) we have

$$\begin{aligned} Y_f(t) = &-\bar{\mathbf{a}}_n Y_{fD}^{(n-1)\alpha}(t) + \bar{\mathbf{b}}_m U_{fD}^{(m-1)\alpha}(t-L) \\ &+ b_0 h F_C^I(t-L) + b_0 h(t-L) + \varsigma(t) \end{aligned} \quad (36)$$

If we define G_f as

$$G_f = \begin{bmatrix} U_{fD}^{(m-1)\alpha}(t-L) \\ hF_C^I(t-L) + ht \end{bmatrix}^T \quad (37)$$

then

$$Y_f(t) = \begin{bmatrix} -Y_{fD}^{(n-1)\alpha}(t) & G_f & -h \end{bmatrix} \begin{bmatrix} \bar{\mathbf{a}}_n \\ \mathbf{b}_m \\ b_0 L \end{bmatrix} + \varsigma(t) \quad (38)$$

or equivalently

$$\psi(t) = \phi(t)\theta + \varsigma(t) \quad (39)$$

where $\theta = \begin{bmatrix} \bar{\mathbf{a}}_\mathbf{n} \\ \mathbf{b}_m \\ b_0 L \end{bmatrix}$.

Similarly, we can write (39) for all $t = t_k$ where $k = t, t+1, ...N$, such that $t > L$, N being the total number of data points. The stacked terms in this equation then yield the following estimation equation

$$\Psi = \Phi\theta + \Delta \quad (40)$$

In practice, the selection of the output $y(t)$ after $t > L$ can be made as follows (Bi et al. (1999)). When the process enters a zero initial state, the process output will be monitored for a period, the listening period, during which the noise band B_n can be found. Then, $y(t)$ satisfying

$$\arg(y(t)) > 2B_n \quad (41)$$

can be treated as the process response after $t > L$, and thus can be used in (40).

3.2 Parameter estimation

When α is known: Since the filter itself involves the coefficients $\mathbf{a}_n$ and we need L in order to formulate the above linear regression equation, we start with some initial values of $\mathbf{a}_n$ and $\mathbf{L}$, then using the above formulation (solving using linear least squares) we can get a new estimate of the parameter vector θ. This parameter vector also gives us updated estimates of $\mathbf{a}_n$(note that $a_0 = 1$) and $\mathbf{L}$. The updated values are again used to get the new estimates. In this way we can iteratively estimate all the model parameters. Note that we still have L term coupled with the b_0 term, so any error in estimating one term translates to another. Now for the case when the data is corrupted with white noise, the filtering operation converts the white noise signal to colored noise and this algorithm gives biased estimates in the presence of colored noise. Thus in order to get the unbiased estimates of the parameters, we use the bootstrap instrumental variable (IV) algorithm (Young (1970)) where the instruments are built based on the auxiliary model (using predicted $y(\hat{y})$ instead of measured y values). The instrument variable is then defined as

$$\phi_{IV}(t) = \begin{bmatrix} -\hat{Y}_{fD}^{(n-1)\alpha}(t) \\ G_f \\ -h \end{bmatrix}^T \quad (42)$$

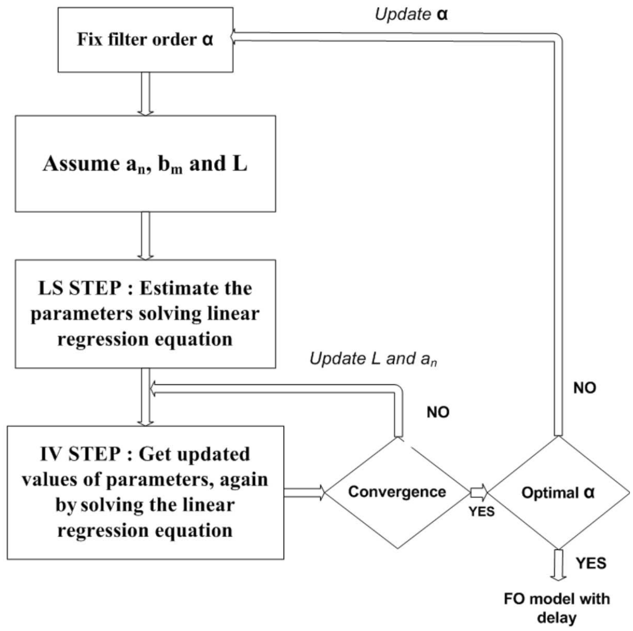

Fig. 1. Algorithm for estimating parameters for CFOTDS

Using this, we can construct the instrumental variable matrix as $\Phi_{IV}(t)$ and we add this IV scheme within the iteration steps of our proposed method thus requiring no additional steps, and the parameter estimation step is then given by

$$\hat{\theta}_{IV}^{(i)} = \left(\Phi_{IV}(\hat{\theta}_{IV}^{(i-1)})^T \Phi(\hat{\theta}_{IV}^{(i-1)})\right)^{-1} \Phi_{IV}(\hat{\theta}_{IV}^{(i)})^T \; \Psi(\hat{\theta}_{IV}^{(i-1)}) \quad (43)$$

where (i) gives the iteration count, and $\Phi_{IV}(\hat{\theta}_{IV}^{(i-1)})$, $\Phi(\hat{\theta}_{IV}^{(i-1)})$ and $\Psi(\hat{\theta}_{IV}^{(i-1)})$ constructs Φ_{IV}, Φ and Ψ respectively for the parameter vector $\hat{\theta}_{IV}^{(i-1)}$.

When α is unknown: In cases when fractional order α is also an unknown variable, we can get a estimate of α by posing the problem as nested loop optimization problem. We start with an initial value of α in the outer loop and in the inner loop we iteratively estimate the model parameters ($\mathbf{a}_n$,$\mathbf{b}_m$) and the delay term (L), as discussed in the previous section and then update α in the outer loop in a non-linear fashion. The algorithm for the proposed scheme is sketched in Fig. (1). The algorithm will not have an outer loop when α is known.

3.3 Summary of the proposed algorithm

The iterative procedure for the parameter estimation can be summarized as

STEP 1. *OUTER LOOP :*

Initialization 1 : *Initialize the algorithm with some initial value for* α

STEP 2. *INNER LOOP :*

Initialization 2 : *Initialize the inner loop with some intial values for* $\hat{\mathbf{a}}_n^{(0)}$ *and* $\hat{L}^{(0)}$

i.) LS Step : $i = 1$

Construct Ψ *and* Φ by replacing $\mathbf{a}_n$ and L with the estimates, as $\hat{\mathbf{a}}_n^{(0)}$ and $\hat{L}^{(0)}$ and get new estimates of parameters as

$$\hat{\theta}^{(1)} = (\Phi^T.\Phi)^{-1}.(\Phi^T.\Psi)$$

Get values of $\hat{\mathbf{a}}_n^{(1)}$,$\hat{\mathbf{b}}_m^{(1)}$ and $\hat{L}^{(1)}$ *from* $\hat{\theta}^{(1)}$.

ii.) IV Step : $i = i + 1$ *to convergence*

Construct Ψ, Φ *and* Φ_{IV} by replacing $\mathbf{a}_n, \mathbf{b}_m$ *and* L with estimates as $\hat{\mathbf{a}}_n^{(i-1)}$, $\hat{\mathbf{b}}_m^{(i-1)}$ and $\hat{L}^{(i-1)}$ and get new $\hat{\theta}^{(i)}$ estimates as

$$\hat{\theta}^{(i)} = (\Phi_{IV}^T.\Phi)^{-1}.(\Phi_{IV}^T.\Psi)$$

Get the values of $\hat{\mathbf{a}}_n^{(i)}$, $\hat{\mathbf{b}}_m^{(i)}$ and $\hat{L}^{(i)}$ from $\hat{\theta}^{(i)}$ and repeat this step till convergence.

STEP 3. *Update value of* α *based on the minimization of the objective function* (*i.e repeat steps* 1 *and* 2 *till this objective function is minimized*)

$$\hat{\alpha} = \arg\min_{\alpha}(\Delta^T\Delta)$$

For the cases when α is known, we will only have the inner loop where the model parameters $\mathbf{a}_n$,$\mathbf{b}_m$ and L are estimated iteratively.

3.4 Convergence issues for the proposed method

The initialization of the inner loop involves choices of $\mathbf{a}_n$,$\mathbf{b}_m$ and L. In practice any initial choice is good except that the filter should not be unstable. As the filter is updated in every step, the final estimate of the parameters is not found to be much sensitive to the initial choice. However, for the outer loop some knowledge on the fractional order is necessary. This is because for some initial values of the fractional order the convergence of inner loop is not always guaranteed. For the case when the fractional order is known, extensive simulation study shows that the parameter estimates obtained in the inner iterative loop converge monotonically to the true parameter values.

4. IDENTIFICATION RESULTS

To illustrate the utility of the proposed algorithm, the identification exercise is carried out on the simplest transfer function of the form given below:

$$G(s) = \frac{b_0}{a_1 s^{\alpha} + 1}\, e^{-Ls} \quad (44)$$

where α is the commensurate fractional order for this model. We performed the identification exercise for both the cases

- When α is known and
- When α is unknown

We performed simulations for the deterministic case and also for the case when a zero mean white noise is added as a disturbance to the system with a signal to noise ratio (SNR) defined by

$$SNR = \frac{var(signal)}{var(noise)} \quad (45)$$

Here, the numerical approximation of (1) is used for simulating the fractional order system.

4.1 Exercise-I : When α *is known*

Example 1: We considered the following fractional order system

$$G_{FO_1}(s) = \frac{1}{5s^{0.5} + 1}\, \mathrm{e}^{-4s} \quad (46)$$

Thus, the true parameters are $b_0 = 1$, $a_1 = 5$, $L = 4$. To illustrate the efficiency of the proposed algorithm the system is simulated using the sampling time of 0.01sec for the deterministic as well as noise case. The output $y(t)$ is corrupted by adding additive Gaussian white noise signal with mean zero and a fixed SNR. Three different values of SNR chosen are ∞(deterministic case), 30 and 10. For each SNR, 100 Monte Carlo (MC) simulations with different noise realizations are performed. For each case we estimated the model parameters (a_1,b_0,L) using the proposed algorithm. Table 1 gives the average ($av(\hat{\theta})$) and the sample standard deviations ($s(\hat{\theta})$) of each parameter for these MC simulations. As can be seen, the estimated parameters are quite close to the true values, thus indicating that the proposed algorithm gives unbiased estimates even in the presence of noise. One interesting thing to note here is that the fractal system when $0 < \alpha < 1$ has very slow dynamics and the step response approaches the new steady state value very slowly and thus in actual practice, one would require infinite amount of data to capture the entire dynamics for this kind of processes. However, the proposed algorithm does well in estimating the parameters for these systems.

Table 1. Estimated parameters for process $G_{FO_1}(s)$ for different SNR

	$SNR = \infty$	$SNR = 30$		$SNR = 10$	
	$\hat{\theta}$	$av(\hat{\theta})$	$s(\hat{\theta})$	$av(\hat{\theta})$	$s(\hat{\theta})$
a_1	5.006	5.004	4.00x10^{-3}	5.000	8.02x10^{-4}
b_0	1.000	1.000	1.82x10^{-4}	1.000	4.82x10^{-5}
L	3.975	3.963	3.73x10^{-2}	3.964	3.58x10^{-2}

Example 2: We considered the following fractional order system

$$G_{FO_2}(s) = \frac{1}{s^{0.75} + 1}\, \mathrm{e}^{-2s} \quad (47)$$

Thus, the true parameters are $b_0 = 1$, $a_1 = 1$, $L = 2$. The sampling time is fixed as 0.01 sec. Again, the identification exercise is performed at SNR values of ∞, 30 and 10 and for each SNR, 100 Monte Carlo (MC) simulations are performed for different noise realizations. For each case we estimated model parameters (a_1,b_0,L) using the proposed algorithm. Table 2 gives the average and the sample standard deviations of each parameter, for these MC simulations. Again, as can be seen the estimated parameters are quite close to the true values, thus indicating that the proposed algorithm gives unbiased estimates in the presence of noise.

4.2 Exercise-II : When α *is unknown*

We considered the same fractional order system as given by (46) for this exercise, however now we are also estimating the commensurate order α:

$$G_{FO_3}(s) = \frac{1}{5s^{0.5} + 1}\, \mathrm{e}^{-4s} \quad (48)$$

Table 2. Estimated parameters for process $G_{FO_2}(s)$ for different SNR

	$SNR = \infty$	$SNR = 30$		$SNR = 10$	
	$\hat{\theta}$	$av(\hat{\theta})$	$s(\hat{\theta})$	$av(\hat{\theta})$	$s(\hat{\theta})$
a_1	0.999	0.999	7.86x10^{-5}	1.001	7.26x10^{-5}
b_0	0.999	0.999	7.26x10^{-5}	0.999	7.26x10^{-5}
L	1.987	1.982	1.81x10^{-2}	1.9815	1.86x10^{-2}

Thus, the true parameters are $b_0 = 1$, $a_1 = 5$, $L = 4$ and $\alpha = 0.50$. The sampling time is fixed as 0.01 sec. The same three values of SNR (∞,30,10) are chosen and for each SNR, 30 Monte Carlo (MC) simulations are performed for different noise realizations. For each case we estimated the fractional order (α) as well as other model parameters (a_1,b_0,L) simultaneously using the proposed nested loop optimization algorithm. Table 3 gives the average and the sample standard deviations of each parameter for these MC simulations. As can be seen the estimated parameters including the fractional order α are quite close to the true values, thus indicating that the proposed algorithm gives unbiased estimates of all the parameters in the presence of noise. However, there are some computational issues with the outer non-linear loop, for some arbitrary guess value of α, the inner loop does not always converge. Therefore, having some process knowledge regarding the fractional order α is important. We started with a initial guess of $\alpha = 0.4$ for all the cases.

Table 3. Estimated parameters for process $G_{FO_3}(s)$ for different SNR

	$SNR = \infty$	$SNR = 30$		$SNR = 10$	
	$\hat{\theta}$	$av(\hat{\theta})$	$s(\hat{\theta})$	$av(\hat{\theta})$	$s(\hat{\theta})$
α	0.501	0.500	2.08x10^{-5}	0.501	4.86x10^{-4}
a_1	5.010	4.995	5.10x10^{-3}	4.995	5.20x10^{-3}
b_0	0.999	0.999	3.38x10^{-4}	0.999	1.10x10^{-3}
L	3.983	3.986	1.45x10^{-2}	3.985	1.53x10^{-2}

5. CONCLUSION

In this paper, continuous-time identification of commensurate fractional order models with time delays is proposed. The proposed method works with step response data. It is based on a linear filtration method where the filter is chosen as a combination of RIVC and a linear integral filter. Using this filter, we can decouple the delay term from other constant model parameters and thus form a linear regression model to estimate these parameters in an iterative manner. For the case when the fractional order α is unknown, a nested loop optimization method is proposed to estimate the time delay along with constant model parameters in an iterative way in the inner loop and the fractional order in the non-linear outer loop. The applicability of the developed procedure is demonstrated on different CFOTDS for the cases when α is known and when α is unknown. In the presence of noise, Monte Carlo simulation analysis for different noise realizations is done to demonstrate that the proposed algorithm gives unbiased estimates even in the presence of noise. In the future, the interesting perspective would be to extend the proposed algorithm for the case where all the parameters are estimated using other types of input excitation signals.

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