Robust Adaptive Dynamic Programming for Large-Scale Systems with an Application to Multimachine Power Systems

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Abstract

This report presents a new approach to decentralized control design of complex systems with unknown parameters and dynamic uncertainties. A key strategy is to use the theory of robust adaptive dynamic programming and the policy iteration technique. An iterative control algorithm is given to devise a decentralized optimal controller that globally asymptotically stabilizes the system in question. Stability analysis is accomplished by means of the small-gain theorem. The effectiveness of the proposed computational control algorithm is demonstrated via the online learning control of multimachine power systems with governor controllers.

Index Terms

Adaptive dynamic programming, decentralized control, small-gain, multimachine power systems.

I. Introduction

In recent years, considerable attention has been paid to the stabilization of large-scale complex systems [7], [17], [21], [22], [23], as well as the related consensus and synchronization problems [1], [16], [24], [34]. Examples of large-scale systems arise from ecosystems, transportation networks, and power systems, to name only a few, [2], [15], [18], [20], [28], [36]. Often, in real-world applications, precise mathematical models are hard to build and the model mismatches, caused by parametric and dynamic uncertainties, are thus unavoidable. This, together with the exchange of only local system information, makes the design problem extremely challenging in the context of complex networks.

The purpose of this paper is to apply the idea of approximate/adaptive dynamic programming (ADP) [29], [30], [31], [32] to find online robust optimal stabilizing controllers for a class of complex large-scale systems. Inspired from the biological learning behavior, ADP is a methodology to solve optimal control problems via online information, without precisely knowing the system dynamics [13], [14], [19], [27]. In our recent work [4], [5], we have integrated modern nonlinear control techniques with ADP to develop a new framework called *robust ADP*. An appealing feature of robust ADP is that dynamic uncertainties can be addressed for the first time for ADP-based computational adaptive controller design.

In this paper, we intend to extend the robust ADP recently developed in [4], [5] to decentralized optimal control of a class of large-scale uncertain systems. The controller design for each subsystem only needs to utilize local state variables without knowing the system dynamics. By integrating a simple version of the cyclic-small-gain theorem [9], asymptotic stability can be achieved by assigning appropriate weighting matrices for each subsystem. As a by-product, certain suboptimality properties can be obtained.

This paper is organized as follows: Section 2 studies the global and robust optimal stabilization of a class of large-scale uncertain systems. Section 3 develops the robust ADP scheme for large-scale systems. Section 4 presents a novel solution to decentralized stabilization based on the proposed methodology. It is our belief that the proposed design methodology will find wide applications in large-scale systems. Finally, Section 5 gives some brief concluding remarks.

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Throughout this paper, we use vertical bars $|\cdot|$ to denote the Euclidean norm for vectors, or the induced matrix norm for matrices. $\otimes$ indicates Kronecker product and $\text{vec}(A)$ represents the mn-vector defined by $\text{vec}(A) = \left[\begin{array}{cccc} a_1^T & a_2^T & \cdots & a_m^T \end{array} \right]^T$, where, for each $i = 1, 2, \cdots, m$, $a_i \in \mathbb{R}^n$ is the i-th column of $A \in \mathbb{R}^{n\times m}$.

II. Optimality and Robustness

In this section, we first describe the class of large-scale uncertain systems to be studied. Then, we present our novel decentralized optimal controller design scheme. It will also be shown that the closed-loop interconnected system enjoys some suboptimality properties.

A. Description of large-scale systems

Consider the complex large-scale system of which the i-subsystem $(1 \le i \le N)$ is described by

$$\dot{x}_i = A_i x_i + B_i \left[u_i + \Psi_i(y)\right], \quad y_i = C_i x_i, \quad 1 \le i \le N \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $y_i \in \mathbb{R}^{q_i}$, and $u_i \in \mathbb{R}^{m_i}$ are the state, the output and the control input for the i-th subsystem; $y = \left[y_1^T, y_2^T, \cdots, y_N^T\right]^T$; $A_i \in \mathbb{R}^{n_i\times n_i}$, $B_i \in \mathbb{R}^{n_i\times m_i}$ are unknown system matrices. $\Psi_i(\cdot) : \mathbb{R}^q \to \mathbb{R}^{m_i}$ are unknown interconnections satisfying $|\Psi_i(y)| \le d_i|y|$ for all $y \in \mathbb{R}^q$, with $d_i > 0$, $\sum_{i=1}^N n_i = n$, $\sum_{i=1}^N q_i = q$, and $\sum_{i=1}^N m_i = m$. It is also assumed that (A_i, B_i) is a stabilizable pair, that is, there exists a constant matrix K_i such that $A_i - B_i K_i$ is a stable matrix.

Notice that the decoupled system can be written in a compact form:

$$\dot{x} = A_D x + B_D u \tag{2}$$

where $x = \left[x_1^T, x_2^T, \cdots, x_N^T\right]^T \in \mathbb{R}^n$, $u = \left[u_1^T, u_2^T, \cdots, u_N^T\right]^T \in \mathbb{R}^m$, $A_D =$block diag$(A_1, A_2, \cdots, A_N) \in \mathbb{R}^{n\times n}$, $B_D =$block diag$(B_1,\ B_2, \cdots, B_N) \in \mathbb{R}^{n\times m}$.

For system (2), we define the following quadratic cost

$$J_D = \int_0^\infty \left(x^T Q_D x + u^T R_D u\right)\ d\tau \tag{3}$$

where $Q_D =$block diag$(Q_1, Q_2, \cdots, Q_N) \in \mathbb{R}^{n\times n}$, $R_D =$block diag$(R_1, R_2, \cdots,\ R_N) \in \mathbb{R}^{m\times m}$, $Q_i \in \mathbb{R}^{n_i\times n_i}$, and $R_i \in \mathbb{R}^{m_i\times m_i}$, with $Q_i = Q_i^T \ge 0$, $R_i = R_i^T > 0$, and $(A_i, Q_i^{1/2})$ observable, for all $1 \le i \le N$.

By linear optimal control theory [12], a minimum cost $J_D^{\odot}$ in (3) can be obtained by employing the following decentralized control policy

$$u_D^{\odot} = -K_D x \tag{4}$$

where $K_D =$ block diag$(K_1, K_2, \cdots, K_N)$ is given by

$$K_D = R_D^{-1} B_D^T P_D \tag{5}$$

and $P_D =$ block diag$(P_1, P_2, \cdots, P_N)$ is the unique symmetric positive definite solution of the algebraic Riccati equation

$$A_D^T P_D + P_D A_D - P_D B_D R_D^{-1} B_D^T P_D + Q_D = 0. \tag{6}$$

B. Decentralized stabilization

Now, we analyze the stability of the closed-loop system comprised of (1) and the decentralized controller (4). We show that by selecting appropriate weighting matrices Q_D and R_D, global asymptotic stability can be achieved for the large-scale closed-loop system.

To begin with, we give two lemmas. Their proofs are given in Appendix A.

Lemma 2.1: For any $\gamma_i > 0$ and $\varepsilon_i > 0$, let $u^{\odot}$ be the decentralized control policy obtained from (4)-(6) with $Q_i \geq \left(\gamma_i^{-1}+1\right) C_i^T C_i + \gamma_i^{-1}\varepsilon_i I_{n_i}$ and $R_i^{-1} \geq d_i^2 I_{m_i}$. Then, along the solutions of the closed-loop system consisting of (1) and (4), we have

$$\frac{d}{dt}\left(x_i^T \gamma_i P_i x_i\right) \leq -|y_i|^2 - \varepsilon_i |x_i|^2 + \gamma_i \sum_{j=1, j\neq i}^{N} |y_j|^2. \tag{7}$$

Lemma 2.2: Under the conditions of Lemma 2.1, suppose the following cyclic-small-gain condition holds

$$\sum_{j=1}^{N-1} j \sum_{1\leq i_1 < i_2 < \cdots < i_{j+1} \leq j+1} \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{j+1}} < 1. \tag{8}$$

Then, there exist constants $c_i > 0$ for all $1 \leq i \leq N$, such that along the solutions of the closed-loop system (1) and (4), we have

$$\frac{d}{dt}\left(\sum_{i=1}^{N} x_i^T c_i \gamma_i P_i x_i\right) \leq -|y|^2 - \sum_{j=1}^{N} c_i \gamma_i \varepsilon_i |x_i|^2. \tag{9}$$

In summary, we obtain the following theorem:

Theorem 2.1: The overall closed-loop system (1), (4) is globally asymptotically stable if the cyclic-small-gain condition (8) holds.

Remark 2.1: It is of interest to note that a more generalized cyclic-small-gain condition based on the notion of input-to-output stability [8], [25] can be found in [9].

C. Suboptimality

Suppose $\Psi_i(\cdot)$ is differentiable at the origin for all $1 \leq i \leq N$, system (1) can be linearized around the origin as

$$\dot{x} = A_D x + B_D u + A_C x, \quad y = C_D x. \tag{10}$$

Notice that under the decentralized control policy (4), the cost (3) yields a minimum cost value $J_D^{\oplus}$ for the coupled system (10), which may differ from $J_D^{\odot}$. In order to study the relationship between $J_D^{\oplus}$ and $J_D^{\odot}$, define

$$M_D = \text{blockdiag}(\mu_1 I_{n_1}, \mu_2 I_{n_2}, \cdots, \mu_N I_{n_N}) \in \mathbb{R}^{n\times n} \tag{11}$$

where $\mu_i > 0$ with $1 \leq i \leq N$.

To quantify the suboptimality of the closed-loop system composed of (10) and (4), we recall the following concept and theorem from [23]:

Definition 2.1 ([23]): The decentralized control law (4) is said to be suboptimal for system (1), if there exists a positive number μ such that

$$J_D^{\oplus} \leq \mu^{-1} J_D^{\odot}. \tag{12}$$

Theorem 2.2 ([23]): Suppose there exists a matrix M_D as defined in (11) such that the matrix

$$F(M_D) = A_C{}^T M_D{}^{-1} P_D + M_D^{-1} P_D A_C + (I - M_D^{-1})(Q_D + K_D^T R_D K_D) \tag{13}$$

satisfy $F(M_D) \leq 0$. Then, the control $u_D^{\odot}$ is suboptimal for (10) with degree

$$\mu = \min_{1 \leq i \leq N} \{\mu_i\}. \tag{14}$$

The following theorem summarizes the suboptimality of the controller (4) under the cyclic-small-gain condition 8. The proof can be found in Appendix A.

Theorem 2.3: The decentralized controller $u_D^{\odot}$ is suboptimal for system (1) with degree

$$\mu = \min_{1 \leq i \leq N} \left\{ \frac{1}{c_i \gamma_i} \min_{1 \leq i \leq N} \left(\frac{c_i \gamma_i}{\gamma_i \varepsilon_i^{-1} \lambda_M + 1}, 1 \right) \right\} \tag{15}$$

if the condition (8) holds.

III. Robust ADP design for large-scale systems

Consider the following algebraic Riccati equation

$$A_i^T P_i + P_i A_i + Q_i - P_i B_i R_i^{-1} B_i^T P_i = 0, \quad 1 \leq i \leq N. \tag{16}$$

It has been shown in [10] that, given $K_i^{(0)}$ such that $A_i - B_i K_i^{(0)}$ is Hurwitz, sequences $\{P_i^{(k)}\}$ and $\{K_i^{(k)}\}$ uniquely determined by

$$\begin{aligned} 0 &= (A_i^{(k)})^T P_i^{(k)} + P_i A_i^{(k)} + Q_i^{(k)}, && (17) \\ K_i^{(k+1)} &= R_i^{-1} B_i^T P_i^{(k)} && (18) \end{aligned}$$

with $A_i^{(k)} = A_i - B_i K_i^{(k)}$, and $Q_i^{(k)} = Q_i + (K_i^{(k)})^T R_i K_i^{(k)}$, have the properties that $\lim_{k \to \infty} P_i^{(k)} = P_i$, $\lim_{k \to \infty} K_i^{(k)} = K_i = R_i^{-1} B_i^T P_i$, and $A_i^{(k)}$ is Hurwitz for all $k \in \mathbb{Z}_+$.

For the i-th subsystem, along the solutions of (1), it follows that

$$\begin{aligned} x_i^T P_i^{(k)} x_i \Big|_t^{t+\delta t} =& 2 \int_t^{t+\delta t} (\hat{u}_i + K_i^{(k)} x_i)^T R_i K_i^{(k+1)} x_i d\tau \\ & - \int_t^{t+\delta t} x_i^T Q_i^{(k)} x_i \; d\tau \end{aligned} \tag{19}$$

where $\hat{u}_i = u_i + \Psi_i(y)$.

For sufficiently large integer $l_i \geq 0$, define $\delta_{xx}^i \in \mathbb{R}^{l_i \times \frac{1}{2} n_i (n_i+1)}$, $I_{xx}^i \in \mathbb{R}^{l_i \times n_i^2}$, and $I_{xu}^i \in \mathbb{R}^{l_i \times m_i n_i}$ as follows

$$\begin{aligned} \delta_{xx}^i &= \begin{bmatrix} \bar{x}_i|_{t_{0,i}}^{t_{1,i}} & \bar{x}_i|_{t_{1,i}}^{t_{2,i}} & \cdots & \bar{x}_i|_{t_{l_i-1,i}}^{t_{l_i,i}} \end{bmatrix}^T, \\ I_{xx}^i &= \begin{bmatrix} \int_{t_{0,i}}^{t_{1,i}} x_i \otimes x_i d\tau & \int_{t_{1,i}}^{t_{2,i}} x_i \otimes x_i d\tau & \cdots & \int_{t_{l_i-1,i}}^{t_{l_i,i}} x_i \otimes x_i d\tau \end{bmatrix}^T, \\ I_{xu}^i &= \begin{bmatrix} \int_{t_{0,i}}^{t_{1,i}} x_i \otimes \hat{u}_i d\tau & \int_{t_{1,i}}^{t_{2,i}} x_i \otimes \hat{u}_i d\tau & \cdots & \int_{t_{l_i-1,i}}^{t_{l_i,i}} x_i \otimes \hat{u}_i d\tau \end{bmatrix}^T \end{aligned}$$

where $\bar{x}_i \in \mathbb{R}^{\frac{1}{2} n_i (n_i+1)}$ is defined as

$$\bar{x}_i = [x_{i,1}^2, x_{i,1} x_{i,2}, \cdots, x_{i,1} x_{i,n_i}, x_{i,2}^2, x_{i,2} x_{i,3}, \cdots, x_{i,n_i-1} x_{i,n_i}, x_{i,n_i}^2]^T$$

and $0 \leq t_{0,i} < t_{1,i} < \cdots < t_{l_i,i}$, for $i = 1, 2, \cdots, N$. Also, for any symmetric matrix $P \in \mathbb{R}^{n_i \times n_i}$, we define $\hat{P} \in \mathbb{R}^{\frac{1}{2} n_i (n_i+1)}$ such that

$$\hat{P} = [p_{11}, 2p_{12}, \cdots, 2p_{1n_i}, p_{22}, 2p_{23}, \cdots, 2p_{n_i-1,n_i}, p_{n_i,n_i}]^T.$$

Then, (19) implies the following matrix form of linear equations

$$\Theta_i^{(k)} \left[\begin{array}{c} \widehat{P}_i^{(k)} \\ \mathrm{vec}(K_i^{(k+1)}) \end{array} \right] = \Xi_i^{(k)} \tag{20}$$

where the matrices $\Theta_i^{(k)} \in \mathbb{R}^{l_i \times \frac{1}{2} n_i(n_i+1)+n_i m_i}$ and $\Xi_i^{(k)} \in \mathbb{R}^{l_i}$ are defined as

$$\Theta_i^{(k)} = \left[\begin{array}{cc} \delta_{xx}^i & -2I_{xx}^i(I_{n_i} \otimes (K_i^{(k)})^T R_i) - 2I_{xu}^i(I_{n_i} \otimes R_i) \end{array} \right],$$
$$\Xi_i^{(k)} = -I_{xx}^i \ \mathrm{vec}(Q_i^{(k)}).$$

Clearly, if (20) has a unique solution, we are able to replace (17) and (18) by (20). In this way, the knowledge of both A_i and B_i is no longer needed.

Assumption 3.1: $\mathrm{rank}\left(\left[I_{xx}^i,\ I_{xu}^i\right]\right) = \frac{n_i(n_i+1)}{2} + n_i m_i$.

Theorem 3.1: Under Assumption 3.1, the matrices $P_i^{(k)} = (P_i^{(k)})^T$ and $K_i^{(k+1)}$ determined by (20) satisfy $\lim_{k\to\infty} P_i^{(k)} = P_i$ and $\lim_{k\to\infty} K_i^{(k)} = K_i$.

See Appendix A for the proof.

In summary, we give the following robust ADP algorithm for practical online implementation. Notice that the algorithm can be implemented to each subsystem in parallel without affecting each other. The learning system implemented for each subsystem only needs to use the state and input information of the subsystem.

Robust ADP algorithm for large-scale systems

1. Select appropriate matrices Q_i and R_i such that the condition (8) is satisfied. $k \leftarrow 0$.
2. For the i-th subsystem, employ $u_i = -K_i^{(0)} x_i + e_i$, with e_i the exploration noise, as the input. Record δ_{xx}^i, I_{xx}^i and I_{xu}^i until Assumption 3.1 is satisfied.
3. Solve $P_i^{(k)}$ and $K_i^{(k+1)}$ from (20).
4. Let $k \leftarrow k+1$, and repeat Step 3 until $|P_i^{(k)} - P_i^{(k-1)}| \le \varepsilon$ for all $k \ge 1$, where the constant $\varepsilon > 0$ can be any predefined small threshold.
5. Use $u_i = -K_i^{(k)} x_i$ as the approximated optimal control policy to the i-th subsystem.

IV. Multimachine Power Systems

A. System model

Consider the classical multimachine power system with governor controllers [11]

$$\dot{\delta}_i(t) = \omega_i(t), \tag{21}$$
$$\dot{\omega}_i(t) = -\frac{D_i}{2H_i}\omega_i(t) + \frac{\omega_0}{2H_i}\left[P_{mi}(t) - P_{ei}(t)\right], \tag{22}$$
$$\dot{P}_{mi}(t) = \frac{1}{T_i}\left[-P_{mi}(t) + u_{gi}(t)\right], \tag{23}$$
$$P_{ei}(t) = E'_{qi}\sum_{j=1}^{N} E'_{qj}\left[B_{ij}\sin\delta_{ij}(t) + G_{ij}\cos\delta_{ij}(t)\right] \tag{24}$$

where $\delta_i(t)$ is the angle of the i-th generator, $\delta_{ij} = \delta_i - \delta_j$; $\omega_i(t)$ is the relative rotor speed; $P_{mi}(t)$ and $P_{ei}(t)$ are the mechanical power and the electrical power; E'_{qi} is the transient EMF in quadrature axis, and is assumed to be constant under high-gain SCR controllers; D_i, H_i, and T_i are the damping constant, the inertia constant and the governor time constant; B_{ij}, G_{ij} are constants for $1 \le i, j \le N$.

Similarly as in [2], system (21)-(23) can be put into the following form,

$$\Delta\dot{\delta}_i(t) = \Delta\omega_i(t), \tag{25}$$

$$\Delta\dot{\omega}_i(t) = -\frac{D_i}{2H_i}\Delta\omega_i(t) + \frac{\omega_0}{2H_i}\Delta P_{mi}(t), \tag{26}$$

$$\Delta\dot{P}_{mi}(t) = \frac{1}{T_i}\left[-\Delta P_{mi}(t) + u_i(t) - d_i(t)\right] \tag{27}$$

where

$$\begin{aligned}
\Delta\delta_i(t) &= \delta_i(t) - \delta_{i0}, \\
\Delta\omega_i(t) &= \omega_i(t) - \omega_{i0}, \\
\Delta P_{mi}(t) &= P_{mi}(t) - P_{ei}(t), \\
u_i(t) &= u_{gi}(t) - P_{ei}(t), \\
d_i(t) &= E'_{qi}\sum_{j=1, j\neq i}^{N} E'_{qj}\left[B_{ij}\cos\delta_{ij}(t) - G_{ij}\sin\delta_{ij}(t)\right]\left[\Delta\omega_i(t) - \Delta\omega_j(t)\right].
\end{aligned}$$

Assume there exists a constant $\beta > 0$ such that $\max_{1\leq i,j\leq N}\left[E'_{qi}E'_{qj}(|B_{ij}| + |G_{ij}|)\right] < \beta$. Then,

$$|d_i(t)| \leq (N-1)\beta\sum_{j=1}^{N}|\Delta\omega_i - \Delta\omega_j| \leq (N-1)^2\beta\sum_{j=1}^{N}|\Delta\omega_j|.$$

Therefore, the model (25)-(27) is in the form (1), if we define $x_i = [\Delta\delta_i(t)\ \Delta\omega_i(t)\ \Delta P_{ei}(t)]^T$ and $y_i = \Delta\omega_i(t)$.

B. Numerical simulation

A ten-machine power system is considered for numerical studies. In the simulation, Generator 1 is used as the reference machine. Governor controllers and ADP-based learning systems are installed on Generators 2-10. Parameters of the system are given in Appendix B. All the parameters, except for the operating points, are assumed to be unknown to the learning system. The weighting matrices are set to be $Q_i = 1000I_3$, $R_i = 1$, for $i = 2, 3, \cdots, 10$.

From $t = 0s$ to $t = 1s$, all the generators were operating at the steady state. At $t = 1s$, an impulse disturbance on the active power was added to the network. As a result, the power angles and frequencies started to oscillate. In order to stabilize the system and improve its performance, the learning algorithm is conducted from $t = 4s$ to $t = 5s$. Robust ADP-based control policies for the generators are applied from $t = 5s$ to the end of the simulation. Trajectories of the angles and frequencies of each generators are shown in Figures 1-6.

V. CONCLUSIONS

The decentralized control of large-scale uncertain systems using robust ADP has been studied in this paper. A novel decentralized controller design scheme is presented. The obtained controller globally asymptotically stabilizes the large-scale system, and at the same time, preserves suboptimality properties. In addition, the effectiveness of the proposed methodology is demonstrated via its application to the online learning control of multimachine power systems with governor controllers.

References

[1] M. Chen, "Some simple synchornoization criteria for complex dynamic networks," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 53, no. 11, 2006.

[2] G. Guo, Y. Wang, and D. J. Hill, "Nonlinear output stabilization control for multimachine power systems," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 47, no. 1, 2000.

[3] Y. Jiang and Z. P. Jiang, "Computational adaptive optimal control for continuous-time linear systems with completely unknown system dynamics," accepted by *Automatica*, 2012.

[4] Y. Jiang and Z. P. Jiang, "Robust adaptive dynamic programming," *Reinforcement Learning and Approximate Dynamic Programming for Feedback Control*, ed. F. L. Lewis and D. Liu, New York: Wiley, 2012.

[5] Y. Jiang and Z. P. Jiang, "Robust approximate dynamic programming and global stabilization with nonlinear dynamic uncertainties," in *Procceedings of the Joint IEEE Conference on Decision and Control and European Control Conference*, pp. 115-120, Orlando, FL, 2011.

[6] Y. Jiang and Z. P. Jiang, "Robust adaptive dynamic programming for large-scale systems with an application to multimachine power systems," *Technical Report*, 2012. Available online: http://files.nyu.edu/yj348/public/papers/2012/tcas12tr.pdf.

[7] Z. P. Jiang, "Decentralized control for large-scale nonlinear systems: a review of recent results," Special Issue in honor of Prof. Šiljak's 70th birthday, *J. Dynamics of Continuous, Discrete and Impulsive Systems*, vol. 11, pp. 537-552, 2004.

[8] Z. P. Jiang, A. R. Teel, and L. Praly, "Small-gain theorem for ISS systems and applications," *Mathematics of Control, Signals, and Systems*, vol. 7, no. 2, pp. 95-120, 1994.

[9] Z. P. Jiang and Y. Wang, "A generalization of the nonlinear small-gain theorem for large-scale complex systems," in *Procceedings of the 7th World Congress on Intelligent Control and Automation*, pp. 1188-1193, 2008.

[10] D. Kleinman, "On an iterative technique for Riccati equation computations," *IEEE Transactions on Automatic Control*, vol. 13, no. 1, pp. 114-115, 1968.

[11] P. Kundur, N. J. Balu, and M. G. Lauby, *Power System Stability and Control*. NY: McGraw-Hill, 1994.

[12] F. L. Lewis and V. L. Syrmos, *Optimal Control*. Wiley, 1995.

[13] F. L. Lewis and D. Vrabie, "Reinforcement learning and adaptive dynamic programming for feedback control," *IEEE Transactions on Circuits and Systems Magazine*, vol. 9, no. 3, pp. 32-50, 2009.

[14] F. L. Lewis, K. G. Vamvoudakis, "Reinforcement learning for partially observable dynamic processes: Adaptive dynamic programming using measured output data," *IEEE Transactions Systems, Man, and Cybernetics, Part B*, vol. 41, no. 1, pp. 14-23, 2011.

[15] Z. Li and G. Chen, "Global synchronization and asymptotic stability of complex dynamic networks," *IEEE Transcations on Circuits and Systems II: Express Briefs*, vol. 53, no. 1, pp. 28-33, 2006.

[16] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 57, no. 1, pp. 213-224, 2010.

[17] A. N. Michel, "On the status of stability of interconnected systems," *IEEE Transactions on Circuits and Systems*, vol. 30, no. 6, pp. 326-340, 1983.

[18] Y. H. Liu, T. J. Chen, C. W. Li, Y. Z. Wang, and B. Chu, "Energy-based L_2 disturbance attenuation excitation control of differential algebraic power systems," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 55, no. 10, pp. 1081-1085, 2008.

[19] J. J. Murray, C. J. Cox, G. G. Lendaris, and R. Saeks, "Adaptive dynamic programming," *IEEE Transactions on Systems, Man, and Cybernetics, Part C: Applications and Reviews*, vol. 32, no. 2, pp. 140-153, 2002.

[20] G. Revel, A. E. León, D. M. Alonso, and J. L. Moiola, "Bifurcation analysis on a multimachine power system model," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 57, no. 4, pp. 937-949, 2010.

[21] N. R. Sandell, P. Varaiya, M. Athans and M. G. Safonov, "Survey of decentralized control methods for large scale systems," *IEEE Transactions on Automatic Control*, vol. 23, pp. 108-128, 1978.

[22] D. D. Šiljak, *Large-scale Dynamic Systems: Stability and Structure*, New York: North-Holland, 1978.

[23] D. D. Šiljak, *Decentralized Control of Complex Systems*, Academic Press, 1991.

[24] Q. Song and J. Cao, "On pinning synchronization of directed and undirected complex dynamical networks," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 57, no. 3, pp. 672-680, 2010.

[25] E. D. Sontag, "Input to state stability: Basic concepts and results," *Nonlinear and Optimal Control Theory,* ed. P. Nistri and G. Stefani, pp. 163-220, Berlin: Springer-Verlag, 2007.

[26] D. Vrabie, O. Pastravanu, M. Abu-Khalaf, and F. L. Lewis, "Adaptive optimal control for continuous-time linear systems based on policy iteration," *Automatica*, vol. 45, no. 2, pp. 477-484, 2009.

[27] F.-Y. Wang, H. Zhang, and D. Liu. "Adaptive dynamic programming: An introduction," *IEEE Computational Intelligence Magazine*, vol. 4, no. 2, pp. 39-47, 2009.

[28] Y. Wang and D. J. Hill, "Robust nonlinear coordinated control of power systems," *Automatica*, vol. 32, no. 4, pp. 611-618, 1995.

[29] P. J. Werbos, *Beyond regression: New tools for prediction and analysis in the behavioural sciences*, PhD Thesis, Harvard University, 1974.

[30] P. J. Werbos, "Neural networks for control and system identification," in *Proceedings of the 28th IEEE Conference on Decision and Control*, pp. 260-265, 1989.

[31] P. J. Werbos, "A menu of designs for reinforcement learning over time," *Neural Networks for Control*, pp. 67-95, ed. W. T. Miller, R. S. Sutton, P. J. Werbos, Cambridge: MIT Press, 1991.

[32] P. J. Werbos, "Approximate dynamic programming for real-time control and neural modeling," *Handbook of Intelligent Control*, ed. D.A. White and D.A. Sofge, New York: Van Nostrand Reinhold, 1992.

[33] P. J. Werbos, "Intelligence in the brain: A theory of how it works and how to build it," *Neural Networks*, vol. 22, no. 3, pp. 200-212, 2009.

[34] X. Yang J. Cao, and J. Lu, "Stochastic synchronization of complex networks with nonidentical nodes via hybrid adaptive and impulsive control," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 59, no. 2, pp. 371-384, 2012.

[35] H. Zhang, Q. Wei, and D. Liu, “An iterative adaptive dynamic programming method for solving a class of nonlinear zero-sum differential games,” *Automatica*, vol. 47, no. 1, pp. 207-214, 2011.
[36] J. Zhou and Y. Ohsawa, “Improved swing equation and its properties in synchronous generators,”, *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 56, no. 1, pp. 200-209, 2009.

APPENDIX A
PROOFS

A. Proof of Lemma 2.1

Along the solutions of the closed-loop system, we have

$$\begin{aligned}
& \frac{d}{dt}\left(x_i^T P_i x_i\right) \\
= \;& x_i^T P_i\left[A_i x_i + B_i u_i + B_i \Psi_i(y)\right] + \left[A_i x_i + B_i u_i + B_i \Psi_i(y)\right]^T P_i x_i \\
= \;& x_i^T P_i\left[A_i x_i - B_i K_i x_i + B_i \Psi_i(y)\right] + \left[A_i x_i - B_i K_i x_i + B_i \Psi_i(y)\right]^T P_i x_i \\
= \;& x_i^T P_i (A_i - B_i K_i) x_i + x_i^T (A_i - B_i K_i)^T P_i x_i + x_i^T P_i B_i \Psi_i(y) + \Psi_i^T(y) B_i^T P_i x_i \\
= \;& x_i^T\left[P_i(A_i - B_i K_i) + (A_i - B_i K_i)^T P_i\right] x_i + x_i^T P_i B_i \Psi_i(y) + \Psi_i^T(y) B_i^T P_i x_i \\
= \;& x_i^T\left[-Q_i - P_i B_i R_i^{-1} B_i^T P_i\right] x_i + x_i^T P_i B_i \Psi_i(y) + \Psi_i^T(y) B_i^T P_i x_i \\
= \;& -x_i^T Q_i x_i - x_i^T P_i B_i R_i^{-1} B_i^T P_i x_i + x_i^T P_i B_i \Psi_i(y) + \Psi_i^T(y) B_i^T P_i x_i \\
\le \;& -x_i^T Q_i x_i - d_i^2 x_i^T P_i B_i B_i^T P_i x_i + x_i^T P_i B_i \Psi_i(y) + \Psi_i^T(y) B_i^T P_i x_i \\
= \;& -x_i^T Q_i x_i + d_i^{-2} \Psi_i^T(y) \Psi_i(y) \\
& - d_i^2 x_i^T P_i B_i B_i^T P_i x_i + x_i^T P_i B_i \Psi_i(y) + \Psi_i^T(y) B_i^T P_i x_i - d_i^{-2} \Psi_i^T(y) \Psi_i(y) \\
= \;& -x_i^T Q_i x_i + d_i^{-2} \Psi_i^T(y) \Psi_i(y) \\
& - \left[d_i B_i^T P_i x_i - d_i^{-1} \Psi_i(y)\right]^T \left[d_i B_i^T P_i x_i - d_i^{-1} \Psi_i(y)\right] \\
\le \;& -x_i^T Q_i x_i + d_i^{-2} \Psi_i^T(y) \Psi_i(y) \\
\le \;& -x_i^T\left[\left(\gamma_i^{-1} + 1\right) C_i^T C_i + \gamma_i^{-1} \varepsilon_i I_{n_i}\right] x_i + d_i^{-2} \Psi_i^T(y) \Psi_i(y) \\
\le \;& -\left(\gamma_i^{-1} + 1\right) |y_i|^2 - \gamma_i^{-1} \varepsilon_i |x_i|^2 + |y|^2 \\
\le \;& -\gamma_i^{-1} |y_i|^2 - \gamma_i^{-1} \varepsilon_i |x_i|^2 + \sum_{j=1, j \ne i}^{N} |y_j|^2 .
\end{aligned}$$

Therefore,

$$\frac{d}{dt}\left(\gamma_i x_i^T P_i x_i\right) \le -|y_i|^2 - \varepsilon_i |x_i|^2 + \gamma_i \sum_{j=1, j \ne i}^{N} |y_j|^2 .$$

The proof is complete.

B. Proof of Lemma 2.2

To begin with, let us consider the following linear equations

$$\begin{bmatrix} -1 & \gamma_2 & \gamma_3 & \cdots & \gamma_N \\ \gamma_1 & -1 & \gamma_3 & \cdots & \gamma_N \\ \gamma_1 & \gamma_2 & -1 & \ddots & \gamma_N \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix} \tag{28}$$

First, we show that, if the cyclic-small-gain condition (8) holds, the equation (28) can be solved as

$$c_i = \frac{\prod_{j=1, j \ne i}^{N} (\gamma_j + 1)}{1 - \sum_{j=1}^{N-1} j \sum_{1 \le i_1 < i_2 < \cdots < i_{j+1} \le j+1} \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_{j+1}}} > 0. \tag{29}$$

Indeed, it can be proved by mathematical induction:

1) if $N=2$, (28) is reduced to

$$\begin{bmatrix} -1 & \gamma_2 \\ \gamma_1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}. \tag{30}$$

and the solution is $c_1 = \dfrac{1+\gamma_2}{1-\gamma_1\gamma_2}$, $c_2 = \dfrac{1+\gamma_1}{1-\gamma_1\gamma_2}$. Notice that the solution is unique, because the cyclic-small-gain condition (8) guarantees that the determinant of the coefficient matrix is non-zero.

2) Suppose (29) is the solution of (28) with $N=N'-1$, we show it is also valid for $N=N'$. Then, from the first row of (28) we have

$$-\prod_{j=1,j\neq i}^{N'-1}(\gamma_j+1)+\sum_{i=2}^{N'-1}\prod_{j=1,j\neq i}^{N'-1}(\gamma_j+1)\gamma_i = \sum_{j=1}^{N'-2} j \sum_{1\le i_1<i_2<\cdots<i_{j+1}\le j+1} \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{j+1}} - 1 \tag{31}$$

Now,

$$\begin{aligned}
& -\prod_{j=1,j\neq i}^{N'}(\gamma_j+1)+\sum_{i=2}^{N'}\prod_{j=1,j\neq i}^{N'}(\gamma_j+1)\gamma_i \\
=\; & -\prod_{j=1,j\neq i}^{N'-1}(\gamma_j+1)(\gamma_{N'}+1)+\sum_{i=2}^{N'-1}\prod_{j=1,j\neq i}^{N'-1}(\gamma_j+1)(\gamma_{N'}+1)\gamma_i+\prod_{j=1}^{N'-1}(\gamma_j+1)\gamma_{N'} \\
=\; & \left[-\prod_{j=1,j\neq i}^{N'-1}(\gamma_j+1)+\sum_{i=2}^{N'-1}\prod_{j=1,j\neq i}^{N'-1}(\gamma_j+1)\gamma_i\right](\gamma_{N'}+1)+\prod_{j=1}^{N'-1}(\gamma_j+1)\gamma_{N'} \\
=\; & \left[\sum_{j=1}^{N'-2} j \sum_{1\le i_1<i_2<\cdots<i_{j+1}\le j+1} \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{j+1}} - 1\right](\gamma_{N'}+1)+\prod_{j=1}^{N'-1}(\gamma_j+1)\gamma_{N'} \\
=\; & \sum_{j=1}^{N'-1} j \sum_{2\le i_1<i_2<\cdots<i_{j+1}\le j+1} \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{j+1}} + \sum_{j=1}^{N'-2} j \sum_{1\le i_1<i_2<\cdots<i_{j+1}\le j+1} \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{j+1}} \\
& +\prod_{j=1}^{N'-1}(\gamma_j+1)\gamma_{N'} - \gamma_{N'} - 1 \\
=\; & \sum_{j=1}^{N'-1} j \sum_{1\le i_1<i_2<\cdots<i_{j+1}\le j+1} \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{j+1}} - 1.
\end{aligned}$$

This implies, with $N=N'$, the first row of (28) is valid with the solution (29). Same derivations can be applied to the rest rows.

Together with Lemma 2.1 we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{i=1}^{N} x_i^T c_i\gamma_i P_i x_i\right)\\
\leq\ & -\sum_{i=1}^{N} c_i\gamma_i\varepsilon_i|x_i|^2+\sum_{i=1}^{N} c_i\left(-|y_i|^2+\gamma_i\sum_{j=1,j\neq i}^{N}|y_j|^2\right)\\
\leq\ & -\sum_{i=1}^{N} c_i\gamma_i\varepsilon_i|x_i|^2+\begin{bmatrix}|y_1|^2\\|y_2|^2\\|y_3|^2\\ \cdots\\|y_N|^2\end{bmatrix}^T\begin{bmatrix}-1&\gamma_2&\gamma_3&\cdots&\gamma_N\\ \gamma_1&-1&\gamma_3&\cdots&\gamma_N\\ \gamma_1&\gamma_2&-1&\ddots&\gamma_N\\ \vdots&\vdots&\ddots&\ddots&\vdots\\ \gamma_1&\gamma_2&\gamma_3&\cdots&-1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\ \vdots\\c_N\end{bmatrix}\\
=\ & -\sum_{j=1}^{N} c_i\gamma_i\varepsilon_i|x_i|^2-|y|^2.
\end{aligned}$$

The proof is complete.

C. *Proof of Theorem 2.1*

Define the Lyapunov candidate

$$V_N=\sum_{i=1}^{N} x_i^T c_i\gamma_i P_i x_i. \tag{32}$$

By Lemma 2.2, along the solutions of (1), it follows

$$\dot{V}_N\leq-\sum_{j=1}^{N} c_i\gamma_i\varepsilon_i|x_i|^2-|y|^2. \tag{33}$$

Hence, the closed-loop system is globally asymptotically stable.

D. *Proof of Theorem 2.3*

Let $\mu_i^{-1}=\alpha c_i\gamma_i$ with $\alpha>\dfrac{1}{\min\limits_{1\leq i\leq N}(c_i\gamma_i)}$ and $\alpha\geq 1$.

Then, by (13) we obtain

$$\begin{aligned}
\frac{d}{dt}\left(x^T M_D^{-1}P_D x\right) &= x^T\left(M_D^{-1}P_D A_D+A_D M_D^{-1}P_D+M_D^{-1}P_D A_C+A_C^T M_D^{-1}P_D\right)x\\
&= x^T\left(-M_D^{-1}Q_D-M_D^{-1}K_D^T R_D K_D+M_D^{-1}P_D A_C+A_C^T M_D^{-1}P_D\right)x\\
&= x^T\left[F(M_D)-Q_D-K_D^T R_D K_D\right]x
\end{aligned}$$

Therefore, by Lemma 2.2, it follows that

$$\begin{aligned} x^T F(M_D)x &= \frac{d}{dt}\left(x^T M_D^{-1} P_D x\right) + x^T\left(Q_D + K_D^T R_D K_D\right)x \\ &\leq \sum_{i=1}^N x_i^T\left[-\mu_i^{-1}(Q_i - C_i^T C_i) - \alpha|y_i|^2 + Q_i + K_i^T R_i K_i\right]x_i \\ &= -\sum_{i=1}^N x_i^T\left[\mu_i^{-1}(Q_i - C_i^T C_i) + \alpha C_i^T C_i - Q_i - K_i^T R_i K_i\right]x_i \\ &= -\sum_{i=1}^N x_i^T\left[\left(\mu_i^{-1} - 1\right)(Q_i - C_i^T C_i) - K_i^T R_i K_i\right]x_i \\ &\leq -\sum_{i=1}^N \left[(\alpha c_i \gamma_i - 1)\frac{\varepsilon_i}{\gamma_i} - \lambda_M\right]|x_i|^2 \end{aligned}$$

where λ_M denotes the maximal eigenvalue of $K_i^T R_i K_i$.

Notice that $F(M_D) \leq 0$, if we set

$$\alpha = \max_{1\leq i\leq N}\left(\frac{\gamma_i \varepsilon_i^{-1}\lambda_M + 1}{c_i\gamma_i}, 1\right) \tag{34}$$

Therefore, we obtain

$$\mu = \min_{1\leq i\leq N}\left\{\frac{1}{c_i\gamma_i}\min_{1\leq i\leq N}\left(\frac{c_i\gamma_i}{\gamma_i\varepsilon_i^{-1}\lambda_M + 1}, 1\right)\right\} \tag{35}$$

The proof is complete by Theorem 2.2.

E. Proof of Theorem 3.1

Step 1): First of all, we show that, for each $i = 1, 2, \cdots, N$, and $k = 0, 1, \cdots$, equation (20) has a unique solution $(P_i^{(k)}, K_i^{(k+1)})$ with $P_i^{(k)} = (P_i^{(k)})^T$.

Notice that it amounts to show that the following linear equation

$$\Theta_i^{(k)} X_i = 0 \tag{36}$$

has only the trivial solution $X_i = 0$, for each $i = 1, 2, \cdots, N$, and $k = 0, 1, \cdots$

To this end, we prove by contradiction: Assume $X_i = \begin{pmatrix} Y_v^i \\ Z_v^i \end{pmatrix} \in \mathbb{R}^{\frac{1}{2}n_i(n_i+1)+m_i n_i}$ is a nonzero solution of (36), where $Y_v^i \in \mathbb{R}^{\frac{1}{2}n_i(n_i+1)}$ and $Z_v^i \in \mathbb{R}^{m_i n_i}$. Then, a symmetric matrix $Y_i \in \mathbb{R}^{n_i\times n_i}$ and a matrix $Z_i \in \mathbb{R}^{m_i\times n_i}$ can be uniquely determined, such that $\hat{Y}_i = Y_v^i$ and $\text{vec}(Z_i) = Z_v^i$.

By (19), we have

$$\Theta_i^{(k)} X_i = I_{xx}^i \text{vec}(M_i) + 2I_{xu}^i \text{vec}(N_i) \tag{37}$$

where

$$M_i = (A_i^{(k)})^T Y + Y A_i^{(k)} + (K_i^{(k)})^T (B_i^T Y_i - R_i Z_i) + (Y_i B_i - Z_i^T R_i) K_i^{(k)}, \tag{38}$$

$$N_i = B_i^T Y_i - R_i Z_i. \tag{39}$$

Notice that since M_i is symmetric, we have

$$I_{xx}^i \text{vec}(M_i) = I_{\bar{x}}^i \hat{M}_i \tag{40}$$

where $I_{\bar{x}}^i \in \mathbb{R}^{l\times \frac{1}{2}n_i(n_i+1)}$ is defined as:

$$I_{\bar{x}}^i = \begin{bmatrix} \int_{t_{0,i}}^{t_{1,i}} \bar{x}_i d\tau, & \int_{t_{1,i}}^{t_{2,i}} \bar{x}_i d\tau, & \cdots, & \int_{t_{l-1,i}}^{t_{l,i}} \bar{x}_i d\tau \end{bmatrix}^T. \tag{41}$$

Then, (36) and (37) imply the following matrix form of linear equations

$$\begin{bmatrix} I_{\bar{x}}^i, & 2I_{xu}^i \end{bmatrix} \begin{bmatrix} \hat{M}_i \\ \mathrm{vec}(N_i) \end{bmatrix} = 0. \tag{42}$$

Under Assumption 3.1, we know $\begin{bmatrix} I_{\bar{x}}^i, & 2I_{xu}^i \end{bmatrix}$ has full column rank. Therefore, the only solution to (42) is $\hat{M}_i = 0$ and $\mathrm{vec}(N_i) = 0$. As a result, we have $M_i = 0$ and $N_i = 0$.

Now, by (39) we know $Z_i = R_i^{-1} B_i^T Y_i$, and (38) is reduced to the following Lyapunov equation

$$(A_i^{(k)})^T Y_i + Y_i A_i^{(k)} = 0. \tag{43}$$

Since $A_i^{(k)}$ is Hurwitz for all $k \in \mathbb{Z}_+$, the only solution to (43) is $Y_i = 0$. Finally, by (39) we have $Z_i = 0$. Also, we have $X_i = 0$. But it contradicts with the assumption that $X_i \neq 0$. Therefore, $\Theta_i^{(k)}$ must have full column rank for all $k \in \mathbb{Z}_+$.

Step 2): Given a stabilizing feedback gain matrix $K_i^{(k)}$, if $P_i^{(k)} = (P_i^{(k)})^T$ is the solution of (17), $K_i^{(k+1)}$ is uniquely determined by $K_i^{(k+1)} = R_i^{-1} B_i^T P_i^{(k)}$. By (19), we know that $P_i^{(k)}$ and $K_i^{(k+1)}$ satisfy (20). On the other hand, let $P = P^T \in \mathbb{R}^{n_i \times n_i}$ and $K \in \mathbb{R}_i^{m_i \times n_i}$, such that

$$\Theta_i^{(k)} \begin{bmatrix} \hat{P} \\ \mathrm{vec}(K) \end{bmatrix} = \Xi_i^{(k)}.$$

Then, we immediately have $\hat{P} = \hat{P}_i^{(k)}$ and $\mathrm{vec}(K) = \mathrm{vec}(K_i^{(k+1)})$. By Step 1), $P = P^T$ and K are unique. In addition, by the definitions of $\hat{P}$ and $\mathrm{vec}(K)$, $P_i^{(k)} = P$ and $K_i^{(k+1)} = K$ are uniquely determined.

Therefore, the policy iteration (20) is equivalent to (17) and (18). By Theorem in [10], the convergence is thus proved.

APPENDIX B
PLANT PARAMETERS

Simulation parameters for the ten-machine power system are shown in Tables 1-3. Also the steady state frequency is set to be $\omega_0 = 314.15\text{rad/s}$. The initial feedback policies are

$$K_i^{(0)} = \begin{bmatrix} 10 & 50 & 0 \end{bmatrix}, \quad 1 \leq i \leq 10. \tag{44}$$

TABLE I
PARAMETERS FOR THE GENERATORS

	G1	G2	G3	G4	G5	G6	G7	G8	G9	G10
$H_i(p.u.)$	∞	6.4	3	5.5	5.2	4.7	5.4	4.9	5.1	3.4
$D_i(p.u.)$	-	1	1.5	2	2.2	2.3	2.6	1.8	1.7	2.9
$T_i(s)$	-	6	6.3	4.9	6.6	5.8	5.9	5.5	5.4	5.5
$E_{qi}(p.u.)$	1	1.2	1.5	0.8	1.3	0.9	1.1	0.6	1.5	1
$\delta_{i0}(^\circ)$	0	108.86	97.4	57.3	68.75	74.48	45.84	68.75	40.11	63.03

The admittance matrices for the transmission lines are

TABLE II
IMAGINARY PARTS OF THE ADMITTANCE MATRIX

B_{ij}	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$
$i=1$	0.2537	0.1875	0.4132	0.2967	0.2852	0.4848	0.2443	0.0908	0.2149	0.2335
$i=2$	0.1875	0.3927	0.2493	0.5291	0.2827	0.2909	0.3759	0.3272	0.2354	0.3819
$i=3$	0.4132	0.2493	0.0545	0.2712	0.2465	0.2230	0.2741	0.2147	0.3280	0.4937
$i=4$	0.2967	0.5291	0.2712	0.5746	0.3255	0.3301	0.1325	0.2878	0.4921	0.1255
$i=5$	0.2852	0.2827	0.2465	0.3255	0.2067	0.3724	0.3049	0.0294	0.2433	0.3146
$i=6$	0.4848	0.2909	0.2230	0.3301	0.3724	0.4621	0.2790	0.4083	0.3542	0.1356
$i=7$	0.2443	0.3759	0.2741	0.1325	0.3049	0.2790	0.1151	0.4265	0.1437	0.5278
$i=8$	0.0908	0.3272	0.2147	0.2878	0.0294	0.4083	0.4265	0.3280	0.1635	0.4432
$i=9$	0.2149	0.2354	0.3280	0.4921	0.2433	0.3542	0.1437	0.1635	0.3644	0.2120
$i=10$	0.2335	0.3819	0.4937	0.1255	0.3146	0.1356	0.5278	0.4432	0.2120	0.3681

TABLE III
REAL PARTS OF THE ADMITTANCE MATRIX

G_{ij}	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$
$i=1$	0.2104	0.0519	0.0005	0.1397	0.0405	-0.0114	0.0009	0.2811	-0.0785	-0.0002
$i=2$	0.0519	0.2940	0.0341	0.0183	0.1294	-0.1749	-0.0035	-0.0051	-0.1086	-0.0372
$i=3$	0.0005	0.0341	0.1520	0.1784	-0.2654	-0.0994	0.1542	-0.0494	0.2254	0.2764
$i=4$	0.1397	0.0183	0.1784	0.0481	-0.1105	-0.0732	0.2210	-0.1121	0.2338	-0.1468
$i=5$	0.0405	0.1294	-0.2654	-0.1105	0.0826	-0.2006	-0.0720	-0.1923	0.0322	-0.1497
$i=6$	-0.0114	-0.1749	-0.0994	-0.0732	-0.2006	0.0232	-0.0289	0.0360	0.1184	-0.0666
$i=7$	0.0009	-0.0035	0.1542	0.2210	-0.0720	-0.0289	0.0390	0.0642	0.0608	-0.0981
$i=8$	0.2811	-0.0051	-0.0494	-0.1121	-0.1923	0.0360	0.0642	0.2687	0.1334	0.1131
$i=9$	-0.0785	-0.1086	0.2254	0.2338	0.0322	0.1184	0.0608	0.1334	0.1934	-0.1729
$i=10$	-0.0002	-0.0372	0.2764	-0.1468	-0.1497	-0.0666	-0.0981	0.1131	-0.1729	0.2053

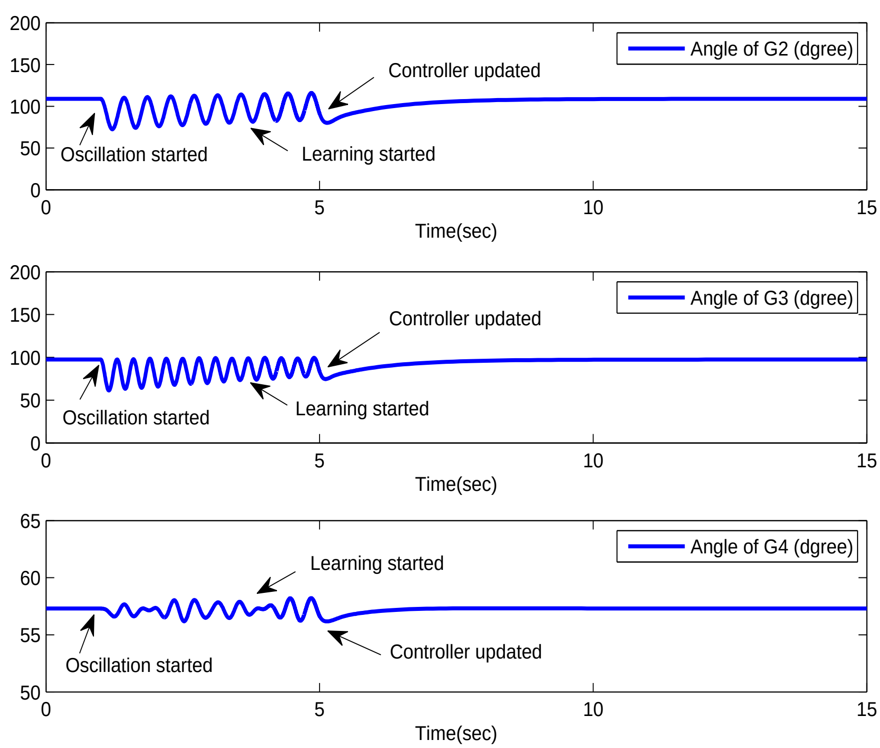

Fig. 1. Power angle deviations of Generators2-4.

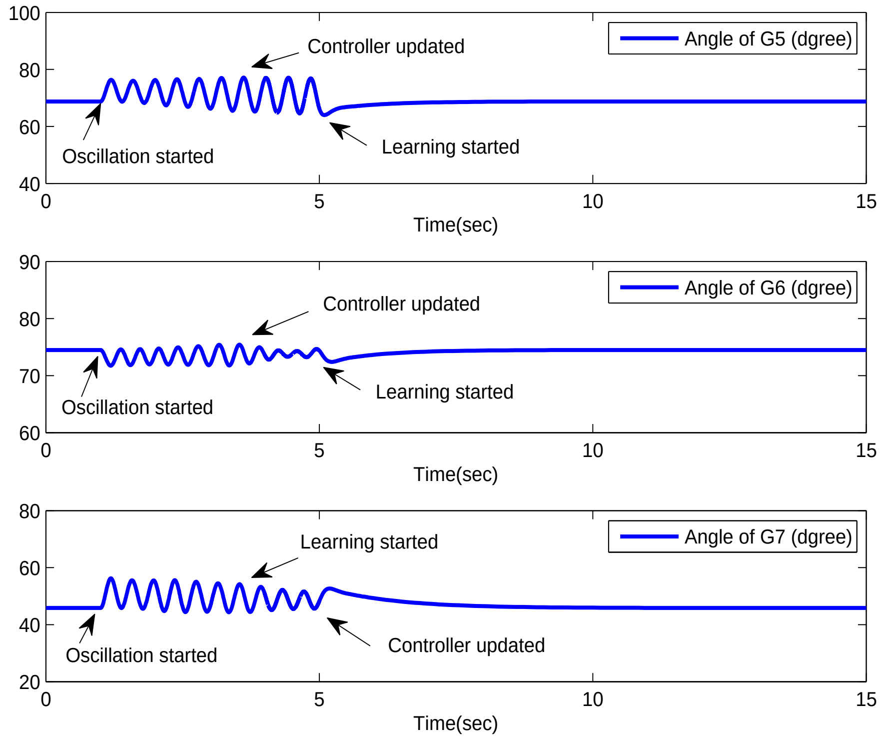

Fig. 2. Power angle deviations of Generators 5-7.

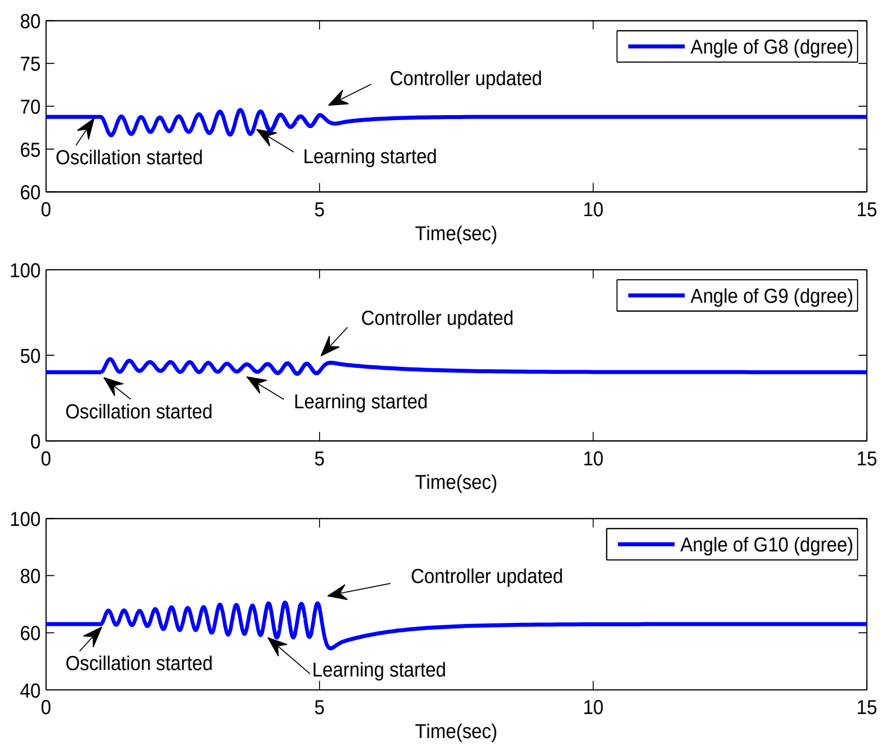

Fig. 3. Power angle deviations of Generators 8-10.

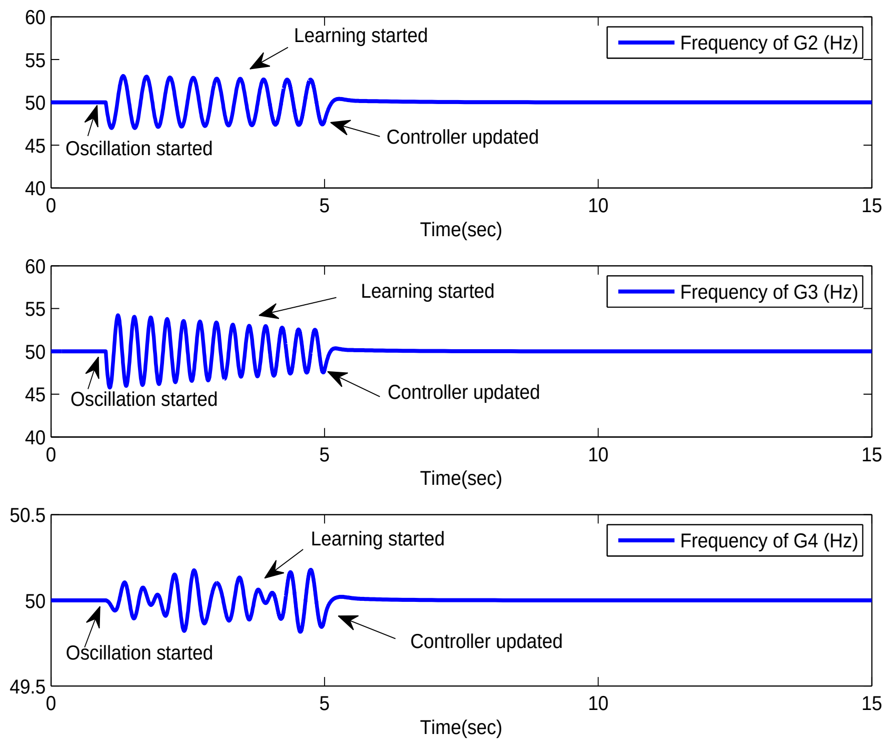

Fig. 4. Power frequencies of Generators 2-4.

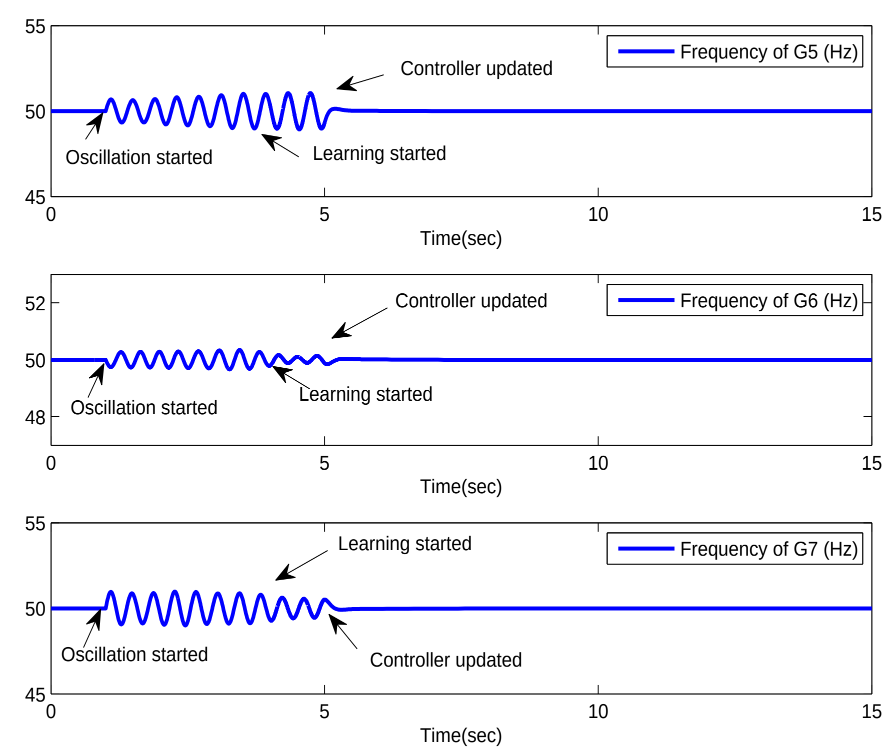

Fig. 5. Power frequencies of Generators 5-7.

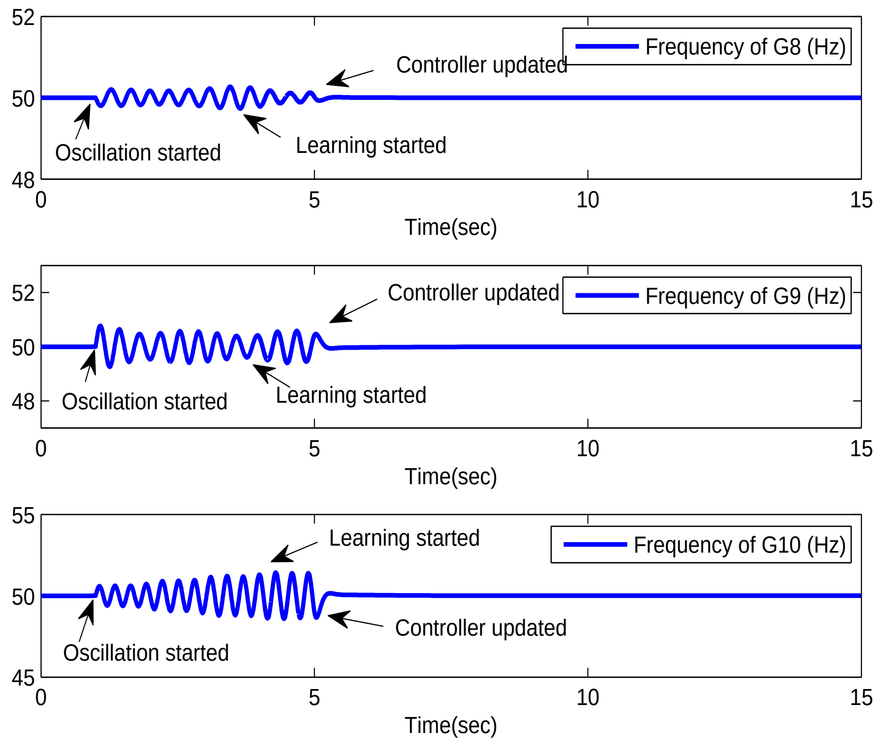

Fig. 6. Power frequencies of Generators 8-10.