Observability Implies Observer Design for Switched Linear Systems

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ABSTRACT

This paper presents a characterization of observability and an observer design method for a class of hybrid systems. A necessary and sufficient condition is presented for observability, globally in time, when the system evolves under predetermined mode transitions. A relatively weaker characterization is given for determinability, the property that concerns with recovery of the original state at some time rather than at all times. These conditions are then utilized in the construction of a hybrid observer that is feasible for implementation in practice. The observer, without using the derivatives of the output, generates the state estimate that converges to the actual state under persistent switching.

Categories and Subject Descriptors

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1. INTRODUCTION

This paper studies observability conditions and observer construction for a class of hybrid systems where the continuous dynamics are modeled as linear differential equations; the state trajectories exhibit jumps during their evolution; and discrete dynamics are represented by an exogenous switching signal. Often called *switched systems*, they are described mathematically as:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \qquad t \neq \{t_q\}, \tag{1a}$$

$$x(t_q) = E_{\sigma(t_q^-)}x(t_q^-) + F_{\sigma(t_q^-)}v_q, \qquad q \geq 1, \tag{1b}$$

$$y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t), \qquad t \geq t_0, \tag{1c}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^{d_y}$ is the output, $v_q \in \mathbb{R}^{d_v}$ and $u(t) \in \mathbb{R}^{d_u}$ are the inputs, and $u(\cdot)$ is a measurable function. The switching signal $\sigma : \mathbb{R} \mapsto \mathbb{N}$ (set of natural numbers) is a piecewise constant and right-continuous function that changes its value at switching times $\{t_q\}$, $q \in \mathbb{N}$. It is assumed that there are a finite number of switching times in any finite time interval, thus we rule out the Zeno phenomenon in our problem formulation. The switching mode $\sigma(t)$ and the switching times $\{t_q\}$ may be governed by a supervisory logic controller, or determined internally depending on the system state, or considered as an external input. In any case, it is assumed in this paper that the signal $\sigma(\cdot)$ (and thus, the active mode and the switching time $\{t_q\}$ as well) is known over the interval of interest. For estimation of the switching signal $\sigma(t)$, one may be referred to, e.g., [4, 7, 13, 14].

In the past decade, the structural properties of hybrid systems have been investigated by many researchers and observability along with observer construction has been one of them. In hybrid systems, the observability can be studied from various perspectives. If we allow for the use of the differential operator in the observer, then it may be desirable to determine the continuous state of the system instantaneously from the measured output. This in turn requires each subsystem to be observable, however, the problem becomes nontrivial when the switching signal is treated as a discrete state and simultaneous recovery of the discrete and continuous state is required for observability. Some results on this problem are published in [2, 6, 13].

On the other hand, if the mode transitions are represented by a known switching signal then, even though the individual subsystems are not observable, it is still possible to recover the initial state $x(t_0)$ when the output is observed over an interval $[t_0, T)$ that involves multiple switching instants. This phenomenon is of particular interest for switched systems as the notion of instantaneous observability and observability over an interval[1] coincide for linear time invariant sys-

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[1]See Definition 1 for precise meaning.

tems. This variant of the observability in switched systems has been studied most notably by [3, 11, 16]. The authors in [8, 9] have studied the observability problem for the systems that allow jumps in the states but they do not consider the change in the dynamics that is introduced by switching to different matrices associated with the active mode. The observer design has also received some attention in the literature [1, 4, 10], where the authors have assumed that each mode in the system is in fact observable admitting a state observer, and have treated the switching as a source of perturbation effect. This approach immediately incurs the need of a common Lyapunov function for the switched error dynamics, or a fixed amount of dwell-time between switching instants, because it is intrinsically a stability problem of the error dynamics.

The approach adopted in this paper is similar to [3, 16] in the sense that we consider observability over an interval. The authors in [3] have presented a coordinate dependent sufficient condition that leads to observer construction; the work of [16] primarily addresses the question whether there exists a switching signal which makes it possible to recover $x(t_0)$ from the knowledge of the output. Whereas, in this paper, similar to our recent work in [12], the switching signal is considered to be known and fixed, so that the trajectory of the system satisfies a time varying linear differential equation. Then for that particular trajectory, we answer the question whether it is possible to recover $x(t_0)$ from the knowledge of the measured output. We present a necessary and sufficient condition for observability over an interval, which is independent of coordinate transformations. Since this condition depends upon the switching times and requires computation of the state transition matrices, we also provide easily verifiable conditions that are either necessary or sufficient for the main condition. Also, with a similar tool set, the notion of determinability, which is more in the spirit of recovering the current state based on the knowledge of inputs and outputs in the past, is developed. Moreover, a hybrid observer for system (1) is designed based on the proposed necessary and sufficient condition which was not the case in [16]. Since the observers are useful for various engineering applications, their utility mainly lies in their online operation method. This thought is essentially rooted in the idea for observer construction adopted in this paper: the idea of combining the partial information available from each mode and collecting them at one instance of time to get the estimate of the state. We show that under mild assumptions, such an estimate converges to the actual state of the plant. We remark that the main contribution of this paper is to present a characterization of observability and an observer design for the systems represented by (1). To the best of authors' knowledge, the considered class of linear systems is the most general for this purpose because it combines both the state jumps and mode switchings.

More emphasis will be given to the case when the individual modes of the system (1) are not observable (in the classical sense of linear time-invariant systems theory) since it is obvious that the system becomes immediately observable when the system is switched to the observable mode. In such cases, switching signal plays a pivotal role as the observability of the switched system not only depends upon the mode sequence but also the switching times. In order to facilitate our understanding of this matter, let us begin with an example.

Example 1. Consider a switched system characterized by:

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

with $E_i = I$, $F_i = 0$, $B_i = 0$, and $D_i = 0$ for $i \in \{1, 2\}$. It is noted that neither of the pair (A_1, C_1) or (A_2, C_2) is observable. However, if the switching signal $\sigma(t)$ changes its value in the order of $1 \to 2 \to 1$ at times t_1 and t_2, then we can recover the state. In fact, it turns out that at least two switchings are necessary and the switching sequence should contain the subsequence of modes $\{1, 2, 1\}$. For instance, if the switching happens as $1 \to 2 \to 1$, the output y at time t_1^- (just before the first switching) and t_2 (just after the second switching) are: $y(t_1^-) = C_1 x(t_1^-) = x_1(t_0)$, and $y(t_2) = C_1 e^{A_2 \tau} x(t_0) = \cos\tau \cdot x_1(t_0) + \sin\tau \cdot x_2(t_0)$, where $x(t_0) = [x_1(t_0), x_2(t_0)]^\top$ is the initial condition and $\tau = t_2 - t_1$. Then, it is obvious that $x(t_0)$ can be recovered from two measurements $y(t_1^-)$ and $y(t_2)$ if $\tau \neq k\pi$ with $k \in \mathbb{N}$. On the other hand, any switching signal whose duration for the mode 2 is an integer multiple of π is a 'singular' switching signal (see Remark 1 for the meaning of singular switching signals).

Notation: For a square matrix A and a subspace $\mathcal{V}$, we denote by $\langle A|\mathcal{V}\rangle$ the smallest A-invariant subspace containing $\mathcal{V}$, and by $\langle \mathcal{V}|A\rangle$ the largest A-invariant subspace contained in $\mathcal{V}$. (See Property 7 in the Appendix for their computation.) With a matrix A, $\mathcal{R}(A)$ denotes the column space (range space) of A. For a possibly non-invertible matrix A, the pre-image of a subspace $\mathcal{V}$ through A is given by $A^{-1}\mathcal{V} = \{x : Ax \in \mathcal{V}\}$. Let $\ker A := A^{-1}\{0\}$, then it is seen that $A^{-1}\ker C = \ker(CA)$ for a matrix C. For convenience of notation, let $A^{-\top}\mathcal{V} := (A^\top)^{-1}\mathcal{V}$ where $A^\top$ is the transpose of A, and it is understood that $A_2^{-1}A_1^{-1}\mathcal{V} = A_2^{-1}(A_1^{-1}\mathcal{V})$. Also, we denote the products of matrices A_i as $\prod_{i=j}^k A_i := A_j A_{j+1} \cdots A_k$ when $j < k$, and $\prod_{i=j}^k A_i := A_j A_{j-1} \cdots A_k$ when $j > k$. The notation $\mathrm{col}(A_1, \ldots, A_k)$ means the vertical stack of matrices $A_1, \cdots, A_k$, that is, $[A_1^\top, \ldots, A_k^\top]^\top$.

2. GEOMETRIC CONDITIONS FOR OBSERVABILITY

To make precise the notions of observability and determinability considered in this paper, let us introduce the formal definitions.

Definition 1. Let $(\sigma^i, u^i, v^i, y^i, x^i)$, for $i = 1, 2$, be the signals that satisfy (1) over an interval[2] $[t_0, T^+)$. We say that the system (1) is $[t_0, T^+)$-*observable* if the equality $(\sigma^1, u^1, v^1, y^1) = (\sigma^2, u^2, v^2, y^2)$ implies that $x^1(t_0) = x^2(t_0)$. Similarly, the system (1) is said to be $[t_0, T^+)$-*determinable*

[2]The notation $[t_0, T^+)$ is used to denote the interval $[t_0, T + \varepsilon)$, where $\varepsilon > 0$ is arbitrarily small. In fact, because of the right continuity of the switching signal, the output $y(T)$ belongs to the next mode when T is the switching instant. Then, the point-wise measurement $y(T)$ is insufficient to contain the information for the new mode, and thus, it is imperative to consider the output signal over the interval $[t_0, T + \varepsilon)$ with $\varepsilon > 0$. This definition implicitly implies that the observability property does not change for sufficiently small ε (which is true, and becomes clear shortly).

if the equality $(\sigma^1, u^1, v^1, y^1) = (\sigma^2, u^2, v^2, y^2)$ implies that $x^1(T) = x^2(T)$.

Since the initial state $x(t_0)$, the switching signal σ, and the inputs (u, v) uniquely determine $x(t)$ on $[t_0, T^+)$ by (1), observability is achieved if and only if the state trajectory $x(t)$, for each $t \in [t_0, T^+)$, is uniquely determined by the inputs, the output, and the switching signal. Obviously, observability implies determinability by forward integration of (1), but the converse is not true due to the possibility of non-invertible matrices E_σ. In case there is no jump map (1b), or each E_σ is invertible, observability and determinability are equivalent. The notion of determinability has also been called reconstructability in [11].

PROPOSITION 1. For a switching signal σ, the system (1) is $[t_0, T^+)$-observable (or, determinable) if, and only if, zero inputs and zero output on the interval $[t_0, T^+)$ imply that $x(t_0) = 0$ (or, $x(T) = 0$).

PROOF. Since the zero solution with the zero inputs yields the zero output, the necessity follows from the fact that $x(t_0)$ (or, $x(T)$) is uniquely determined from the inputs and the outputs. For the sufficiency, suppose that the system (1) is not $[t_0, T^+)$-observable (or determinable); that is, there exist two different states $x^1(t_0)$ and $x^2(t_0)$ (or, $x^1(T)$ and $x^2(T)$) that yield the same output y under the same inputs (u, v). Let $\tilde{x}(t) := x^1(t) - x^2(t)$, where $x^i(t)$, $i = 1, 2$, is the solution of (1) which takes the value $x^i(t_0)$ at initial time t_0 (or, $x^i(T)$ at terminal time T). Then, by linearity, it follows that $\dot{\tilde{x}} = A_\sigma \tilde{x}$, $\tilde{x}(t_q) = E_\sigma \tilde{x}(t_q^-)$, and $C_\sigma \tilde{x} = C_\sigma x^1 - C_\sigma x^2 = y - y = 0$, but $\tilde{x}(t_0) = x^1(t_0) - x^2(t_0) \neq 0$ (or, $\tilde{x}(T) = x^1(T) - x^2(T) \neq 0$). Hence, zero inputs and zero output do not imply $x(t_0) = 0$ (or, $x(T) = 0$), and the sufficiency holds. $\square$

Because of Proposition 1, we are motivated to introduce the following homogeneous switched system, which has been obtained by setting the inputs (u, v) equal to zero in (1):

$$\dot{x}(t) = A_{\sigma(t)} x(t), \quad y(t) = C_{\sigma(t)} x(t), \quad t \in [t_{q-1}, t_q) \tag{2a}$$

$$x(t_q) = E_{\sigma(t)} x(t_q^-). \tag{2b}$$

If this homogeneous system is observable (or, determinable) with a given σ, then $y \equiv 0$ implies that $x(t_0) = 0$ (or, $x(T) = 0$), and in terms of description of system (1), this means that zero inputs and zero output yield $x(t_0) = 0$ (or, $x(T) = 0$); hence, (1) is observable (or, determinable) because of Proposition 1. On the other hand, if the system (1) is observable (or, determinable), then it is still observable (or, determinable) with zero inputs, which is described as system (2). Thus, the observability (or, determinability) of systems (1) and (2) are equivalent.

Before going further, let us rename the switching sequence for convenience. For system (1), when the switching signal $\sigma(t)$ takes the mode sequence $\{q_1, q_2, q_3, \cdots\}$, we rename them as increasing integers $\{1, 2, 3, \cdots\}$, which is ever increasing even though the same mode is revisited; for convenience, this sequence is indexed by q and not by $\sigma(t)$. Moreover, it is often the case that the mode of the system changes without the state jump (1b), or the state jumps without switching to another mode. In the former case, we can simply take $E_q = I$ and $F_q = 0$, and in the latter case, we increase the mode index by one and take $A_q = A_{q+1}$, $B_q = B_{q+1}$ and so on. In this way various situations fit into the description of (1) with increasing mode sequence. The switching time t_q is the instant when transition from mode q to mode $q+1$ takes place.

2.1 Necessary and Sufficient Conditions for Observability

In this section, we present a characterization of the unobservable subspace for the system (2) with a given switching signal. Towards this end, let $\mathcal{N}_q^m$ $(m \geq q)$ denote the set of states at $t = t_{q-1}$ for system (2) that generate identically zero output over $[t_{q-1}, t_{m-1}^+)$. Then, it is easily seen that $\mathcal{N}_q^m$ is actually a subspace due to linearity of (2), and we call $\mathcal{N}_q^m$ the *unobservable subspace for* $[t_{q-1}, t_{m-1}^+)$. It can be seen that the system (2) is an LTI system between two consecutive switching times, so that its unobservable subspace on the interval $[t_{q-1}, t_q)$ is simply given by the largest A_q-invariant subspace contained in $\ker C_q$, i.e., $\langle \ker C_q | A_q \rangle = \ker G_q$ where $G_q := \mathrm{col}(C_q, C_q A_q, \cdots, C_q A_q^{n-1})$. So it is clear that $\mathcal{N}_q^q = \ker G_q$. Now, when the measured output is available over the interval $[t_{q-1}, t_{m-1}^+)$ that includes switchings at $t_q, t_{q+1}, \ldots, t_{m-1}$, more information about the state is obtained in general so that $\mathcal{N}_q^m$ gets smaller as the difference $m - q$ gets larger, and we claim that the subspace $\mathcal{N}_q^m$ can be computed recursively as follows:

$$\begin{aligned} \mathcal{N}_m^m &= \ker G_m, \\ \mathcal{N}_q^m &= \ker G_q \cap e^{-A_q \tau_q} E_q^{-1} \mathcal{N}_{q+1}^m, \quad 1 \leq q \leq m-1, \end{aligned} \tag{3}$$

where $\tau_q = t_q - t_{q-1}$. The following theorem presents a necessary and sufficient condition for observability of the system (1) while proving the claim in the process.

THEOREM 1. For the system (2) with a switching signal $\sigma_{[t_0, t_{m-1}^+)}$, the unobservable subspace for $[t_0, t_{m-1}^+)$ at t_0 is given by $\mathcal{N}_1^m$ from (3). Therefore, the system (1) is $[t_0, t_{m-1}^+)$-observable if, and only if,

$$\mathcal{N}_1^m = \{0\}. \tag{4}$$

From (3), it is not difficult to arrive at the following formula for $\mathcal{N}_q^m$:

$$\mathcal{N}_q^m = \ker G_q \cap \left(\bigcap_{i=q+1}^{m} \prod_{j=q}^{i-1} e^{-A_j \tau_j} E_j^{-1} \ker G_i \right) \tag{5a}$$

$$= \ker G_q \cap \left(\bigcap_{i=q+1}^{m} \ker \left(G_i \prod_{l=i-1}^{q} E_l e^{A_l \tau_l} \right) \right). \tag{5b}$$

From this expression, it is easily seen that $\mathcal{N}_1^{m_1} \supseteq \mathcal{N}_1^{m_2}$ if $m_1 \leq m_2$. Therefore, in case the interval under consideration is not finite and the switching is persistent, observability of system (1) is determined by whether there exists an $m \in \mathbb{N}$ such that (4) holds.

PROOF OF THEOREM 1. *Sufficiency.* Using the result of Proposition 1, it suffices to show that the identically zero output of (2) implies $x(t_0) = 0$. Assume that $y \equiv 0$ on $[t_0, t_{m-1}^+)$. Then, it is immediate that $x(t_{m-1}) \in \mathcal{N}_m^m = \ker G_m$. We next apply the inductive argument to show that $x(t_{q-1}) \in \mathcal{N}_q^m$ for $1 \leq q \leq m-1$. Suppose that $x(t_q) \in \mathcal{N}_{q+1}^m$, then $x(t_{q-1}) \in e^{-A_q \tau_q} E_q^{-1} \mathcal{N}_{q+1}^m$ since $x(t)$

is the solution of (2). Zero output on the interval $[t_{q-1}, t_q)$ also implies that $x(t_{q-1}) \in \ker G_q$. Therefore,

$$x(t_{q-1}) \in \ker G_q \cap e^{-A_q\tau_q} E_q^{-1} \mathcal{N}_{q+1}^m.$$

From (3), it follows that $x(t_{q-1}) \in \mathcal{N}_q^m$. This induction proves the claim that $\mathcal{N}_q^m$ is given by (3). With $q = 1$, it is seen that $x(t_0) \in \mathcal{N}_1^m = \{0\}$, which proves the sufficiency.

Necessity. Assuming that $\mathcal{N}_1^m \neq \{0\}$, we show that a non-zero initial state $x(t_0) \in \mathcal{N}_1^m$ yields the solution of (2) such that $y \equiv 0$ on $[t_0, t_{m-1}^+)$, which implies unobservability. First, we show the following implication;

$$x(t_{q-1}) \in \mathcal{N}_q^m \quad \Rightarrow \quad x(t_q) \in \mathcal{N}_{q+1}^m, \qquad q < m. \qquad (6)$$

Indeed, assuming that $x(t_{q-1}) \in \mathcal{N}_q^m$ with $q < m$, it follows that, $x(t_q) = E_q e^{A_q\tau_q} x(t_{q-1})$, which further gives,

$$\begin{aligned} x(t_q) &\in E_q e^{A_q\tau_q} \mathcal{N}_q^m \\ &= E_q e^{A_q\tau_q} \left(\ker G_q \cap e^{-A_q\tau_q} E_q^{-1} \mathcal{N}_{q+1}^m\right) \\ &\subseteq E_q \ker G_q \cap E_q E_q^{-1} \mathcal{N}_{q+1}^m \\ &= E_q \ker G_q \cap \mathcal{N}_{q+1}^m \cap \mathcal{R}(E_q) \subseteq \mathcal{N}_{q+1}^m \end{aligned}$$

by using (3) and Properties 2, 3, and 11 in the Appendix. Therefore, for $0 \le q \le m-1$, $x(t_q) \in \mathcal{N}_{q+1}^m \subseteq \ker G_{q+1}$, and the solution $x(t) = e^{A_{q+1}(t-t_q)} x(t_q)$ for $t \in [t_q, t_{q+1})$ satisfies that $y(t) = C_{q+1}x(t) = 0$ for $t \in [t_q, t_{q+1})$ due to A_{q+1}-invariance of $\ker G_{q+1}$. □

Remark 1. The observability condition (4) given in Theorem 1 is dependent upon a particular switching signal under consideration, and it is entirely possible that the system is observable for certain switching signals and unobservable for others (*cf.* Example 1). Note that a switching signal is composed of a mode sequence and switching times. We call a switching signal σ *singular* when the observability condition (4) does not hold with σ, but the condition happens to hold by changing the switching times of σ while preserving the mode sequence.

In order to inspect the observability of the system (2), one can compute $\mathcal{N}_1^m$ by (5) (the formula (5b) may be preferable because the computation of pre-image due to E_j^{-1} is avoided). However, the computation of matrix exponent may be heavy in practice (especially for large dimensional systems) and one may want to resort to the following sufficient, or necessary conditions, which are independent of switching times and only take mode sequence into consideration. Hence, once the sufficient condition in Corollary 1 holds (respectively, the necessary condition in Corollary 2 is violated), then the system is observable (resp. unobservable) for any switching signal that has the same switching mode sequence regardless of the switching times.

Corollary 1. Let $\overline{\mathcal{N}}_1^m$ be an over-approximation of $\mathcal{N}_1^m$ that is defined as follows:

$$\begin{aligned} \overline{\mathcal{N}}_m^m &:= \ker G_m, \\ \overline{\mathcal{N}}_q^m &:= \left\langle A_q \middle| \ker G_q \cap E_q^{-1} \overline{\mathcal{N}}_{q+1}^m \right\rangle, \qquad 1 \le q \le m-1. \end{aligned}$$

The system (1) is $[t_0, t_{m-1}^+)$-observable if $\overline{\mathcal{N}}_1^m = \{0\}$.

Proof. The proof is completed by showing that $\mathcal{N}_q^m \subseteq \overline{\mathcal{N}}_q^m$ for $1 \le q \le m$. First, note that $\mathcal{N}_m^m = \overline{\mathcal{N}}_m^m$. Assuming that $\mathcal{N}_{q+1}^m \subseteq \overline{\mathcal{N}}_{q+1}^m$ for $1 \le q \le m-1$, we now claim that $\mathcal{N}_q^m \subseteq \overline{\mathcal{N}}_q^m$. Indeed, by Properties 3, 9, and 11 in the Appendix, and the recursion equation (3), we obtain

$$\begin{aligned} \mathcal{N}_q^m &= \ker G_q \cap e^{-A_q\tau_q} E_q^{-1} \mathcal{N}_{q+1}^m \\ &= e^{-A_q\tau_q} \left(\ker G_q \cap E_q^{-1} \mathcal{N}_{q+1}^m\right) \\ &\subseteq \left\langle A_q \middle| \ker G_q \cap E_q^{-1} \mathcal{N}_{q+1}^m \right\rangle \\ &\subseteq \left\langle A_q \middle| \ker G_q \cap E_q^{-1} \overline{\mathcal{N}}_{q+1}^m \right\rangle = \overline{\mathcal{N}}_q^m, \quad 1 \le q \le m-1. \end{aligned}$$

Therefore, the condition $\overline{\mathcal{N}}_1^m = \{0\}$ implies (4). □

Corollary 2. Let $\underline{\mathcal{N}}_1^m$ be an under-approximation of $\mathcal{N}_1^m$ that is defined as follows:

$$\begin{aligned} \underline{\mathcal{N}}_m^m &:= \ker G_m, \\ \underline{\mathcal{N}}_q^m &:= \left\langle \ker G_q \cap E_q^{-1} \underline{\mathcal{N}}_{q+1}^m \middle| A_q \right\rangle, \qquad 1 \le q \le m-1. \end{aligned}$$

If system (1) is $[t_0, t_{m-1}^+)$-observable, then $\underline{\mathcal{N}}_1^m = \{0\}$.

Proof. The proof proceeds similar to Corollary 1. With $\mathcal{N}_m^m = \underline{\mathcal{N}}_m^m$, we assume that $\mathcal{N}_{q+1}^m \supseteq \underline{\mathcal{N}}_{q+1}^m$ for $1 \le q \le m-1$, and claim that $\mathcal{N}_q^m \supseteq \underline{\mathcal{N}}_q^m$. Again by Properties 3, 9, and 11 in the Appendix, and employing equation (3), we obtain

$$\begin{aligned} \mathcal{N}_q^m &= e^{-A_q\tau_q} \left(\ker G_q \cap E_q^{-1} \mathcal{N}_{q+1}^m\right) \\ &\supseteq \left\langle \ker G_q \cap E_q^{-1} \mathcal{N}_{q+1}^m \middle| A_q \right\rangle \\ &\supseteq \left\langle \ker G_q \cap E_q^{-1} \underline{\mathcal{N}}_{q+1}^m \middle| A_q \right\rangle = \underline{\mathcal{N}}_q^m, \quad 1 \le q \le m-1. \end{aligned}$$

The condition $\underline{\mathcal{N}}_1^m = \{0\}$ is implied by (4). □

Remark 2. By taking orthogonal complements of $\mathcal{N}_q^m$, $\overline{\mathcal{N}}_q^m$ and $\underline{\mathcal{N}}_q^m$, respectively, we get dual conditions, using Properties 5, 6, 8, and 10 in the Appendix, as follows. The system (1) is $[t_0, t_{m-1}^+)$-observable if and only if $\mathcal{P}_1^m = \mathbb{R}^n$ where

$$\mathcal{P}_1^m := (\mathcal{N}_1^m)^\perp = \mathcal{R}(G_1^\top) + \sum_{i=2}^{m} \prod_{j=1}^{i-1} e^{A_j^\top \tau_j} E_j^\top \mathcal{R}(G_j^\top).$$

Based on the above definition, one can state Corollary 1 and Corollary 2 in alternate forms. System (1) is $[t_0, t_{m-1}^+)$-observable if $\underline{\mathcal{P}}_1^m = \mathbb{R}^n$, where $\underline{\mathcal{P}}_1^m$ is computed as:

$$\begin{aligned} \underline{\mathcal{P}}_m^m &= (\overline{\mathcal{N}}_m^m)^\perp = \mathcal{R}(G_m^\top) \\ \underline{\mathcal{P}}_q^m &= (\overline{\mathcal{N}}_q^m)^\perp = \left\langle \mathcal{R}(G_q^\top) + E_q^\top \underline{\mathcal{P}}_{q+1}^m \middle| A_q^\top \right\rangle, \quad 1 \le q \le m-1. \end{aligned}$$

Also, if system (1) is $[t_0, t_{m-1}^+)$-observable then $\overline{\mathcal{P}}_1^m = \mathbb{R}^n$, where $\overline{\mathcal{P}}_1^m$ is defined sequentially as:

$$\begin{aligned} \overline{\mathcal{P}}_m^m &= (\underline{\mathcal{N}}_m^m)^\perp = \mathcal{R}(G_m^\top) \\ \overline{\mathcal{P}}_q^m &= (\underline{\mathcal{N}}_q^m)^\perp = \left\langle A_q^\top \middle| \mathcal{R}(G_q^\top) + E_q^\top \overline{\mathcal{P}}_{q+1}^m \right\rangle, \quad 1 \le q \le m-1. \end{aligned}$$

2.2 Necessary and Sufficient Conditions for Determinability

In order to study determinability of the system (1) and arrive at a result parallel to Theorem 1, our first goal is to develop an object similar to $\mathcal{N}_q^m$. So, for system (2) with a given switching signal, let $\mathcal{Q}_q^m$ be the set of states that can be reached at time $t = t_{m-1}$ while producing the zero output on the interval $[t_{q-1}, t_{m-1}^+)$. We call $\mathcal{Q}_q^m$ the *undeterminable subspace for* $[t_{q-1}, t_{m-1}^+)$. Then, it can be shown, similarly to the proof of Theorem 1, that $\mathcal{Q}_q^m$ is computed recursively as follows:

$$\begin{aligned}
&\mathcal{Q}_q^q = \ker G_q \\
&\mathcal{Q}_q^k = \ker G_k \cap E_{k-1} e^{A_{k-1}\tau_{k-1}} \mathcal{Q}_q^{k-1}, \quad q+1 \le k \le m.
\end{aligned} \tag{7}$$

These sequential definitions lead to following expression for $\mathcal{Q}_q^m$:

$$\begin{aligned}
\mathcal{Q}_q^m = \ker G_m \cap E_{m-1} \ker(G_{m-1}) \cap \\
\left(\bigcap_{i=q}^{m-2} \prod_{l=m-1}^{i+1} E_l e^{A_l \tau_l} E_i \ker G_i \right),
\end{aligned} \tag{8}$$

with $\mathcal{Q}_q^q = \ker G_q$. In the above equation, the subspace $(\Pi_{l=m-1}^{i+1} E_l e^{A_l\tau_l} E_i \ker G_i)$ indicates the set of states at time $t = t_{m-1}$ obtained by propagating the unobservable state of the mode i, that is active during the interval $[t_{i-1}, t_i)$, under the dynamics of system (2). Intersection of these subspaces with $\ker G_m$ shows that $\mathcal{Q}_q^m$ is the set of states that cannot be determined from the zero output at time $t = t_{m-1}$. Then, the determinability can be characterized as in the following theorem (which is given without proof).

Theorem 2. For the system (2) and a given switching signal $\sigma_{[t_0, t_{m-1}^+)}$, the undeterminable subspace for $[t_0, t_{m-1}^+)$ at t_{m-1} is given by $\mathcal{Q}_1^m$ of (8). Therefore, the system (1) is $[t_0, t_{m-1}^+)$-determinable if and only if

$$\mathcal{Q}_1^m = \{0\}. \tag{9}$$

The condition (9) is equivalent to (4) when all E_q matrices, $q = 1, \dots, m-1$, are invertible because of the relation

$$\mathcal{Q}_1^m = \prod_{l=m-1}^{1} E_l e^{A_l \tau_l} \mathcal{N}_1^m.$$

On the other hand, if any of the jump maps E_q is a zero matrix, then (9) holds regardless of (4) (which makes sense because we can immediately determine that $x(t_{m-1}) = 0$ in this case).

The recursive expression (7) shows that the sequence $\{\mathcal{Q}_1^1, \mathcal{Q}_1^2, \mathcal{Q}_1^3, \cdots\}$ is moving forward in the sense that $\mathcal{Q}_1^{k+1}$ is computed from $\mathcal{Q}_1^k$ and from the information about the running mode k such as G_{k+1}, E_k, A_k, and τ_k. This fact illustrates that the computation of $\mathcal{Q}_1^m$ is more suitable for online implementation (since m increases as time sets forward), compared to the computation of $\mathcal{N}_1^m$, which requires a backward computation from $\mathcal{N}_m^m$ (see (3)).

Corollary 3. The system (1) is $[t_0, t_{m-1}^+)$-determinable if $\overline{\mathcal{Q}}_1^m = \{0\}$, where $\overline{\mathcal{Q}}_1^m$ is computed by

$$\begin{aligned}
&\overline{\mathcal{Q}}_1^1 := \ker G_1 \\
&\overline{\mathcal{Q}}_1^q := E_{q-1} \left\langle A_{q-1} | \overline{\mathcal{Q}}_1^{q-1} \right\rangle \cap \ker G_q, \quad 2 \le q \le m.
\end{aligned}$$

Corollary 4. If system (1) is $[t_0, t_{m-1}^+)$-determinable, then $\underline{\mathcal{Q}}_1^m = \{0\}$, where $\underline{\mathcal{Q}}_1^m$ is computed by

$$\begin{aligned}
&\underline{\mathcal{Q}}_1^1 := \ker G_1 \\
&\underline{\mathcal{Q}}_1^q := E_{q-1} \left\langle \underline{\mathcal{Q}}_1^{q-1} | A_{q-1} \right\rangle \cap \ker G_q, \quad 2 \le q \le m.
\end{aligned}$$

The above corollaries are proved by showing that $\underline{\mathcal{Q}}_1^q \subseteq \mathcal{Q}_1^q \subseteq \overline{\mathcal{Q}}_1^q$. It is noted again that the computation of sequential subspaces in Corollary 3 and Corollary 4 proceeds forward in time.

Remark 3. An alternative dual characterization of determinability is possible by inspecting whether the complete state information is available while going forward in time. This is achieved in terms of the subspace $\mathcal{M}_q^m$, obtained by taking the orthogonal complement of $\mathcal{Q}_q^m$. Using Properties 5, 6, 8, and 10 in the Appendix, the following expression follows from (8):

$$\begin{aligned}
\mathcal{M}_q^m := (\mathcal{Q}_q^m)^\perp = \sum_{i=q}^{m-2} \prod_{l=m-1}^{i+1} E_l^{-\top} e^{-A_l^\top \tau_l} E_i^{-\top} \mathcal{R}(G_i^\top) \\
+ E_{m-1}^{-\top} \mathcal{R}(G_{m-1}^\top) + \mathcal{R}(G_m^\top).
\end{aligned} \tag{10}$$

In other words, $\mathcal{M}_q^m$ is the set of states at time instant $t = t_{m-1}$ that can be identified, modulo the unobservable subspace at t_{m-1}, from the information of y over the interval $[t_{q-1}, t_{m-1}^+)$. Therefore, the dual statement for determinability is that the system (1) is $[t_0, t_{m-1}^+)$-determinable if and only if

$$\mathcal{M}_1^m = \mathbb{R}^n. \tag{11}$$

It is noted that a recursive expression for $\mathcal{M}_1^m$ is given by

$$\begin{aligned}
&\mathcal{M}_1^1 = \mathcal{R}(G_1^\top) \\
&\mathcal{M}_1^q = E_{q-1}^{-\top} e^{-A_{q-1}^\top \tau_{q-1}} \mathcal{M}_1^{q-1} + \mathcal{R}(G_q^\top), \quad 2 \le q \le m,
\end{aligned}$$

and the dual statements of Corollaries 3 and 4, that are independent of switching times, are given as follows: system (1) is $[t_0, t_{m-1}^+)$-determinable if $\underline{\mathcal{M}}_1^m = \mathbb{R}^n$, where

$$\begin{aligned}
&\underline{\mathcal{M}}_1^1 := (\overline{\mathcal{Q}}_1^m)^\perp = \mathcal{R}(G_1^\top), \\
&\underline{\mathcal{M}}_1^q := (\overline{\mathcal{Q}}_1^q)^\perp = E_{q-1}^{-\top} \left\langle \underline{\mathcal{M}}_1^{q-1} | A_{q-1}^\top \right\rangle + \mathcal{R}(G_q^\top), \ 2 \le q \le m.
\end{aligned}$$

Similarly, if system (1) is $[t_0, t_{m-1}^+)$-determinable then $\overline{\mathcal{M}}_1^m = \mathbb{R}^n$, where $\overline{\mathcal{M}}_1^m$ is computed as follows:

$$\begin{aligned}
&\overline{\mathcal{M}}_1^1 := (\underline{\mathcal{Q}}_1^m)^\perp = \mathcal{R}(G_1^\top), \\
&\overline{\mathcal{M}}_1^q := (\underline{\mathcal{Q}}_1^q)^\perp = E_{q-1}^{-\top} \left\langle A_{q-1}^\top | \overline{\mathcal{M}}_1^{q-1} \right\rangle + \mathcal{R}(G_q^\top), \ 2 \le q \le m.
\end{aligned}$$

3. OBSERVER DESIGN

In engineering practice, an observer is designed to provide an estimate of the actual state value at current time. In this regard, determinability (weaker than observability according to Definition 1) is a suitable notion. Based on the conditions obtained for determinability in the previous section, an asymptotic observer is designed for the system (1) in this section. By asymptotic observer, we mean that the estimate $\hat{x}(t)$ converges to the plant state $x(t)$ as $t \to \infty$, and in order to achieve this convergence, we introduce the following assumptions.

Assumption 1. 1. The switching is persistent in the sense that there exists a $D > 0$ such that a switch occurs at least once in every time interval of length D; that is,

$$t_q - t_{q-1} < D, \qquad \forall\, q \in \mathbb{N}. \tag{12}$$

2. The system is persistently determinable in the sense that there exists an $N \in \mathbb{N}$ such that

$$\dim \mathcal{M}_{q-N}^{q} = n, \qquad \forall\, q \geq N+1. \tag{13}$$

(The integer N is interpreted as the minimal number of switches required to gain determinability.)

3. $\|A_q\|$ is uniformly bounded for all $q \in \mathbb{N}$ (which is always the case when A_q belongs to a finite set).

We disregard the time consumed for computation by assuming that the data processor is fairly fast compared to the plant process. The computation time, however, needs to be considered in real-time application if the plant itself is fast.

The observer we propose is a hybrid dynamical system of the form

$$\dot{\hat{x}}(t) = A_q\hat{x}(t) + B_q u(t), \qquad t \in [t_{q-1}, t_q), \tag{14a}$$

$$\hat{x}(t_q) = E_q(\hat{x}(t_q^-) - \xi_q(t_q^-)) + F_q v_q, \quad q \geq 1, \tag{14b}$$

$$\xi_q(t_q^-) = \begin{cases} \mathcal{L}_q(y_{[t_{q-N-1},t_q)}, u_{[t_{q-N-1},t_q)}, v_{[q-N,q-1]}), & q > N, \\ 0, & 1 \leq q \leq N, \end{cases} \tag{14c}$$

with an arbitrary initial state $\hat{x}(t_0) \in \mathbb{R}^n$, where $v_{[q-N,q-1]}$ denotes the vector $[v_{q-N}, v_{q-N+1}, \ldots, v_{q-1}]^\top$. It is seen that the observer consists of a system copy and an estimate update law by some operator $\mathcal{L}_q$. So the goal is to design the operator $\mathcal{L}_q$ such that $\hat{x}(t) \to x(t)$. It will turn out that the proposed operator $\mathcal{L}_q$ consists of dynamic observers for partial states at each mode, a procedure for accumulating the partial state information, and an inversion formula for recovering full state information. In the sequel, we develop the structure of the operator $\mathcal{L}_q$ and based on that, a procedure for implementation of hybrid observer is outlined in Algorithm 1. It is then shown in Theorem 3 that the state estimates computed according to the parameter bounds in Algorithm 1 indeed converge to the actual state of the system.

With $\tilde{x} := \hat{x} - x$, the error dynamics are described by,

$$\dot{\tilde{x}}(t) = A_q\tilde{x}(t), \qquad t \neq t_q, \tag{15a}$$

$$\tilde{x}(t_q) = E_q(\tilde{x}(t_q^-) - \xi_q(t_q^-)). \tag{15b}$$

The output error can now be defined as $\tilde{y}(t) := C_q\hat{x}(t) + D_q u(t) - y(t) = C_q\tilde{x}(t)$.

Based on the description of error dynamics, we design partial observers for each mode q using the idea similar to Kalman observability decomposition [5]. Choose a matrix Z^q such that its columns are an orthonormal basis of $\mathcal{R}(G_q^\top)$, so that $\mathcal{R}(Z^q) = \mathcal{R}(G_q^\top)$. Further, choose a matrix W^q such that its columns are an orthonormal basis of $\ker G_q$. From the construction, there are matrices $S_q \in \mathbb{R}^{r_q \times r_q}$ and $R_q \in \mathbb{R}^{d_y \times r_q}$, where $r_q = \operatorname{rank} G_q$, such that $Z^{q\top} A_q = S_q Z^{q\top}$ and $C_q = R_q Z^{q\top}$, and that the pair (S_q, R_q) is observable. Let $z^q := Z^{q\top}\tilde{x}$ and $w^q := W^{q\top}\tilde{x}$, so that z^q (resp. w^q) denotes the observable (resp. unobservable) states of mode q. Thus, for the interval $[t_{q-1}, t_q)$, we obtain,

$$\dot{z}^q = Z^{q\top} A_q \tilde{x} = S_q z^q, \quad \tilde{y} = C_q\tilde{x} = R_q z^q, \tag{16a}$$

$$z^q(t_{q-1}) = Z^{q\top}\tilde{x}(t_{q-1}). \tag{16b}$$

Since z^q is observable over the interval $[t_{q-1}, t_q)$, a standard Luenberger observer, whose role is to estimate $z^q(t_q^-)$ at the end of the interval, is designed as:

$$\dot{\hat{z}}^q = S_q\hat{z}^q + L_q(\tilde{y} - R_q\hat{z}^q), \quad t \in [t_{q-1}, t_q), \tag{17a}$$

$$\hat{z}^q(t_{q-1}) = 0, \tag{17b}$$

where L_q is a matrix such that $(S_q - L_q R_q)$ is Hurwitz. Note that we have fixed the initial condition of the estimator to be zero for each interval.

Let us denote the vector $[\tau_{i+1}, \cdots, \tau_j]$ simply by $\tau_{\{i+1,j\}}$ (where $j > i$), which will be often dropped when used as an argument for succinct presentation. With $j > i$, define the state-transport matrix

$$\Psi_i^j(\tau_{\{i+1,j\}}) := e^{A_j\tau_j} E_{j-1} e^{A_{j-1}\tau_{j-1}} E_{j-2} \cdots e^{A_{i+1}\tau_{i+1}} E_i, \tag{18}$$

and for convenience $\Psi_i^i := I$. We now define a matrix $\Theta_i^q(\tau_{\{i+1,q\}})$ whose columns form a basis of the subspace $\mathcal{R}(\Psi_i^q(\tau_{\{i+1,q\}})W^i)^\perp$; that is,

$$\mathcal{R}(\Theta_i^q(\tau_{\{i+1,q\}})) = \mathcal{R}(\Psi_i^q(\tau_{\{i+1,q\}})W^i)^\perp, \quad i = q-N, \cdots, q.$$

By construction, each column of Θ_i^q is orthogonal to the subspace $\ker G_i$ that has been transported from t_i^- to t_q^- along the error dynamics (15). This matrix Θ_i^q will be used for filtering out the unobservable component in the state estimate obtained from the mode i after being transported to the time t_q^-. As a convention, we take Θ_i^q to be a null matrix whenever $\mathcal{R}(\Psi_i^q(\tau_{\{i+1,q\}})W^i)^\perp = \{0\}$.

Using the determinability of the system (Assumption 1.2), it will be shown later in the proof of Theorem 3 that the matrix

$$\Theta_q := [\Theta_q^q \,\vdots\, \cdots \,\vdots\, \Theta_{q-N}^q] \tag{19}$$

has rank n. Equivalently, $\Theta_q^\top$ has n independent columns and is left-invertible, so that $(\Theta_q^\top)^\dagger = (\Theta_q\Theta_q^\top)^{-1}\Theta_q$, where $\dagger$ denotes the left-pseudo-inverse. Introduce the notation

$$\begin{aligned} \xi_{\{q-N,q-1\}}^- &:= \operatorname{col}(\xi_{q-N}(t_{q-N}^-), \ldots, \xi_{q-1}(t_{q-1}^-)), \\ \hat{z}_{\{q-N,q\}}^- &:= \operatorname{col}(\hat{z}^{q-N}(t_{q-N}^-), \ldots, \hat{z}^q(t_q^-)), \end{aligned} \tag{20}$$

and define the vector Ξ_q as follows:

$$\Xi_q(\hat{z}_{\{q-N,q\}}^-, \xi_{\{q-N,q-1\}}^-) := \begin{bmatrix} \Theta_q^{q\top}\Psi_q^q Z^q \hat{z}^q(t_q^-) \\ \Theta_{q-1}^{q\top}\left(\Psi_{q-1}^q Z^{q-1}\hat{z}^{q-1}(t_{q-1}^-) - \Psi_{q-1}^q\xi_{q-1}(t_{q-1}^-)\right) \\ \vdots \\ \Theta_{q-N}^{q\top}\left(\Psi_{q-N}^q Z^{q-N}\hat{z}^{q-N}(t_{q-N}^-) - \sum_{l=q-N}^{q-1}\Psi_l^q\xi_l(t_l^-)\right) \end{bmatrix}.$$

We then compute $\xi_q(t_q^-)$ in (14c) as:

$$\xi_q(t_q^-) = (\Theta_q^\top)^\dagger \Xi_q(\hat{z}_{\{q-N,q\}}^-, \xi_{\{q-N,q-1\}}^-) \tag{21}$$

which corresponds to the operator $\mathcal{L}_q$. Finally, as the last piece of notation, we define the matrices M_j^q, $j = q-N, \cdots, q$, as follows:

$$\begin{aligned} &[M_q^q, M_{q-1}^q, \cdots, M_{q-N}^q] := E_q(\Theta_q^\top)^\dagger \times \\ &\quad \operatorname{blockdiag}\left(\Theta_q^{q\top}\Psi_q^q, \Theta_{q-1}^{q\top}\Psi_{q-1}^q, \cdots, \Theta_{q-N}^{q\top}\Psi_{q-N}^q\right). \end{aligned} \tag{22}$$

Each M_j^q ($j = q-N, \cdots, q$) is an n by n matrix whose argument is $\tau_{\{q-N+1,q\}}$ in general (due to the inversion of $\Theta_q^\top$), while the argument of both Θ_j^q and Ψ_j^q is $\tau_{\{j+1,q\}}$.

With all the quantities defined so far, the proposed observer (14) is implemented using the following algorithm:

Algorithm 1 Implementation of hybrid observer

Input: σ, u, v, y

Initialization: Run (14) for $t \in [t_0, t_{N+1})$ with some $\hat{x}(t_0)$

1: **for all** $q \geq N+1$ **do**

2: **for** $j = q-N$ **to** q **do**

3: Compute the injection gain L_j such that

$$\|M_j^q(\tau_{\{q-N+1,q\}}) Z^j e^{(S_j - L_j R_j)\tau_j} Z^{j\top}\| \leq c \quad (23)$$

where the constant c is chosen so that

$$0 < c < \frac{1}{N+1}. \quad (24)$$

4: Obtain $\hat{z}^j(t_j^-)$ by running the individual observer (17) for the j-th mode.

5: **end for**

6: Compute $\xi_q(t_q^-)$ from (21), as an implementation of (14c).

7: Compute $\hat{x}(t_q)$ using (14b) and run (14a) over the interval $[t_q, t_{q+1})$.

8: **end for**

The following theorem shows that the above implementation guarantees the convergence of the estimation error to zero.

Theorem 3. Under Assumption 1, if the hybrid observer (14) is implemented as in Algorithm 1, with $\mathcal{L}_q$ computed through the observers (17) and the map $(\Theta_q^\top)^\dagger \Xi_q$ in (21), then the estimated state $\hat{x}(t)$ has the property that:

$$\lim_{t\to\infty} |\hat{x}(t) - x(t)| = 0. \quad (25)$$

Remark 4. Note that the computation of the gains L_j's requires the knowledge of switching times in order to generate converging estimates. Thus, post-processing of the data is required since L_j's cannot be computed a priori. Also, in the operation of the observer, we ignored the time required for computation at time instant t_q. In fact, the outcome $\xi_q(t_q^-)$ becomes available not at t_q but at $t_q + T_{comp}$ where T_{comp} is the time elapsed during computation. It is conjectured that the error caused by this time-delayed update in (14) can be suppressed by taking smaller value of c in (24) while the update is actually performed at $t_q + T_{comp}$ using another state-transport matrix. Detailed analysis on improving the quality of the observer is an ongoing work.

Proof of Theorem 3. Using (15), it follows from Assumptions 1.1 and 1.3 that the estimation error $\tilde{x}(t)$ for the interval $[t_q, t_{q+1})$ is bounded by

$$|\tilde{x}(t)| = |e^{A_{q+1}(t-t_q)}\tilde{x}(t_q)| \leq e^{a(t-t_q)}|\tilde{x}(t_q)|$$

with a constant a such that $\|A_q\| \leq a$ for all $q \in \mathbb{N}$, and thus,

$$|\tilde{x}(t)| \leq e^{aD}|\tilde{x}(t_q)|.$$

Therefore, if $|\tilde{x}(t_q)| \to 0$ as $q \to \infty$, then we achieve that

$$\lim_{t\to\infty} |\tilde{x}(t)| = 0. \quad (26)$$

The remainder of the proof shows that $|\tilde{x}(t_q)| \to 0$ as $q \to \infty$ under the conditions stated in the theorem statement.

Note that, $\tilde{x}(t_q^-)$ can be written as,

$$\tilde{x}(t_q^-) = \begin{bmatrix} Z^{q\top} \\ W^{q\top} \end{bmatrix}^{-1} \begin{bmatrix} z^q(t_q^-) \\ w^q(t_q^-) \end{bmatrix} = Z^q z^q(t_q^-) + W^q w^q(t_q^-). \quad (27)$$

The matrix $\Psi_i^j(\tau_{\{i+1,j\}})$, defined in (18), transports $\tilde{x}(t_i^-)$ to $\tilde{x}(t_j^-)$ along (15) by

$$\tilde{x}(t_j^-) = \Psi_i^j(\tau_{\{i+1,j\}})\tilde{x}(t_i^-) - \sum_{l=i}^{j-1} \Psi_l^j(\tau_{\{l+1,j\}})\xi_l(t_l^-). \quad (28)$$

We now have the following series of equivalent expressions for $\tilde{x}(t_q^-)$:

$$\begin{aligned} \tilde{x}(t_q^-) &= Z^q z^q(t_q^-) + W^q w^q(t_q^-) \\ &= \Psi_{q-1}^q Z^{q-1} z^{q-1}(t_{q-1}^-) + \Psi_{q-1}^q W^{q-1} w^{q-1}(t_{q-1}^-) \\ &\qquad - \Psi_{q-1}^q \xi_{q-1}(t_{q-1}^-) \\ &= \Psi_{q-2}^q Z^{q-2} z^{q-2}(t_{q-2}^-) + \Psi_{q-2}^q W^{q-2} w^{q-2}(t_{q-2}^-) \\ &\qquad - \Psi_{q-2}^q \xi_{q-2}(t_{q-2}^-) - \Psi_{q-1}^q \xi_{q-1}(t_{q-1}^-) \\ &\ \ \vdots \\ &= \Psi_{q-N}^q Z^{q-N} z^{q-N}(t_{q-N}^-) \\ &\quad + \Psi_{q-N}^q W^{q-N} w^{q-N}(t_{q-N}^-) - \sum_{l=q-N}^{q-1} \Psi_l^q \xi_l(t_l^-). \end{aligned} \quad (29)$$

To appreciate the implication of this equivalence, we first note that for each $q-N \leq i \leq q$, the term $\Psi_i^q Z^i z^i(t_i^-)$ transports the observable information of the i-th mode from the interval $[t_{i-1}, t_i)$ to the time instant t_q^-. This observable information is corrupted by the unknown term $w^i(t_i^-)$, but since the information is being accumulated at t_q^- from modes $i = q-N, \cdots, q$, the idea is to combine the partial information from each mode to recover $\tilde{x}(t_q^-)$. This is where we use the notion of determinability. By Properties 1, 5, and 6 in the Appendix, and the fact that $\mathcal{R}(W^i)^\perp = (\ker G_i)^\perp = \mathcal{R}(G_i^\top)$ and $e^{-A_q^\top \tau_q}\mathcal{R}(G_q^\top) = \mathcal{R}(G_q^\top)$, it follows under Assumption 1.2 that

$$\begin{aligned} &\mathcal{R}(W^q)^\perp + \mathcal{R}(\Psi_{q-1}^q W^{q-1})^\perp + \cdots + \mathcal{R}(\Psi_{q-N}^q W^{q-N})^\perp \\ &= e^{-A_q^\top \tau_q}\Big(\mathcal{R}(G_q^\top) + E_{q-1}^{-\top}\mathcal{R}(G_{q-1}^\top) + \\ &\qquad \sum_{i=q-N}^{q-2} \Pi_{l=q-1}^{i+1} E_l^{-\top} e^{-A_l^\top \tau_l} E_i^{-\top}\mathcal{R}(G_i^\top)\Big) \\ &= e^{-A_q^\top \tau_q}\mathcal{M}_{q-N}^q = \mathbb{R}^n. \end{aligned}$$

This equation shows that the matrix Θ_q defined in (19) has rank n, and is left-invertible. Keeping in mind that the range space of each Θ_i^q is orthogonal to $\mathcal{R}(\Psi_i^q W^i)$, each equality in (29) leads to the following relation:

$$\Theta_i^{q\top}\tilde{x}(t_q^-) = \Theta_i^{q\top}\left(\Psi_i^q Z^i z^i(t_i^-) - \sum_{l=i}^{q-1} \Psi_l^q \xi_l(t_l^-)\right), \quad (30)$$

for $i = q-N, \cdots, q$. Stacking (30) from $i = q$ to $i = q-N$, and employing the left-inverse of $\Theta_q^\top$, we obtain that

$$\tilde{x}(t_q^-) = (\Theta_q^\top)^\dagger \Xi_q(z_{\{q-N,q\}}^-, \xi_{\{q-N,q-1\}}^-) \quad (31)$$

where $z^-_{\{q-N,q\}}$ is defined similarly as in (20). It is seen from (31) that, if we are able to estimate $z^-_{\{q-N,q\}}$ without error, then the plant state $x(t_q^-)$ is exactly recovered by (31) because $x(t_q^-) = \hat{x}(t_q^-) - \tilde{x}(t_q^-)$ and both entities on the right side of the equation are known. However, since this is not the case, $z^-_{\{q-N,q\}}$ has been replaced with its estimate $\hat{z}^-_{\{q-N,q\}}$ in (21), and $\xi_q(t_q^-)$ is set as an estimate of $\tilde{x}(t_q^-)$ there.

Thanks to the linearity of Ξ_q in its arguments, it is noted that,

$$\begin{aligned}\tilde{x}(t_q) &= E_q(\tilde{x}(t_q^-) - \xi_q(t_q^-)) \\ &= E_q(\Theta_q^\top)^\dagger\Big(\Xi_q(z^-_{\{q-N,q\}}, \xi^-_{\{q-N,q-1\}}) \\ &\qquad - \Xi_q(\hat{z}^-_{\{q-N,q\}}, \xi^-_{\{q-N,q-1\}})\Big) \\ &= -E_q(\Theta_q^\top)^\dagger \Xi_q(\tilde{z}^-_{\{q-N,q\}}, 0)\end{aligned} \tag{32}$$

where $\tilde{z}^-_{\{q-N,q\}} := \hat{z}^-_{\{q-N,q\}} - z^-_{\{q-N,q\}} = \mathrm{col}(\tilde{z}^{q-N}(t^-_{q-N}), \ldots, \tilde{z}^q(t_q^-))$. It follows from (16) and (17) that

$$\tilde{z}^i(t_{i-1}) = \hat{z}^i(t_{i-1}) - z^i(t_{i-1}) = 0 - Z^{i\top}\tilde{x}(t_{i-1})$$

and

$$\tilde{z}^i(t_i^-) = e^{(S_i - L_i R_i)\tau_i}\tilde{z}^i(t_{i-1}) = -e^{(S_i - L_i R_i)\tau_i} Z^{i\top}\tilde{x}(t_{i-1}).$$

Plugging this expression in (32), and using the definition of M_j^q ($j = q-N, \ldots, q$) from (22), we get

$$\tilde{x}(t_q) = \sum_{j=q-N}^{q} M_j^q(\tau_{\{q-N+1,q\}}) Z^j e^{(S_j - L_j R_j)\tau_j} Z^{j\top}\tilde{x}(t_{j-1}).$$

Then, from the selection of gains L_j's satisfying (23), we have that

$$|\tilde{x}(t_q)| \le \sum_{j=q-N}^{q} c|\tilde{x}(t_{j-1})|. \tag{33}$$

Finally, the statement of the following lemma, proof of which appears in the Appendix, aids us in the completion of the proof of Theorem 3

Lemma 1. A sequence $\{a_i\}$ satisfying

$$|a_i| \le c(|a_{i-1}| + |a_{i-2}| + \cdots + |a_{i-N-1}|), \qquad i > N,$$

with $0 \le c < 1/(N+1)$ converges to zero: $\lim_{i\to\infty} a_i = 0$.

Applying Lemma 1 to (33), we see that $|\tilde{x}(t_q)| \to 0$ as $q \to \infty$, whence the desired result follows. □

Example 2. We demonstrate the operation of the proposed observer for the switched system considered in Example 1. We assume that each mode is activated for τ seconds, so that the persistent switching signal is:

$$\sigma(t) = \begin{cases} 1 & \text{if } t \in [2k\tau, (2k+1)\tau), \\ 2 & \text{if } t \in [(2k+1)\tau, (2k+2)\tau), \end{cases} \tag{34}$$

where $k = 0, 1, 2, \cdots$, and the underlying assumption is that $\tau \ne \kappa\pi$ for any $\kappa \in \mathbb{N}$. As mentioned earlier, the system is observable (and thus, determinable) with this switching signal if the mode sequence $1 \to 2 \to 1$ is contained in a time interval. Hence, we pick $N = 3$ for Assumption 1.2 to hold. For brevity, we call $[2k\tau, (2k+1)\tau)$, the *odd* interval, and $[(2k+1)\tau, (2k+2)\tau)$, the *even* interval. With an arbitrary initial condition $\hat{x}(0)$, the observer to be implemented is:

$$\left.\begin{aligned}\dot{\hat{x}}(t) &= A_1\hat{x}(t) \\ \hat{y}(t) &= C_1\hat{x}(t)\end{aligned}\right\}, \quad t \in [2k\tau, (2k+1)\tau), \tag{35a}$$

$$\left.\begin{aligned}\dot{\hat{x}}(t) &= A_2\hat{x}(t) \\ \hat{y}(t) &= C_2\hat{x}(t)\end{aligned}\right\}, \quad t \in [(2k+1)\tau, (2k+2)\tau), \tag{35b}$$

$$\hat{x}(q\tau) = \hat{x}(q\tau^-) - \xi_q(q\tau^-), \qquad q > 3. \tag{35c}$$

In order to determine the value of $\xi_q(q\tau^-)$, we start off with the estimators for the observable modes of each subsystem, denoted by z^q in (16). Note that mode 1 has a one-dimensional observable subspace whereas for mode 2, the unobservable subspace is $\mathbb{R}^2$. Since mode 1 is active on every odd interval and mode 2 on every even interval, z^q for every odd q represents the partial information obtained from mode 1, and z^q for every even q is a null vector as no information is extracted from mode 2. So the one-dimensional z-observer in (17) is implemented only for odd intervals. For odd q, we compute

$$G_q = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \mathcal{R}(G_q^\top) = \mathrm{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}, W^q = \begin{bmatrix}0\\1\end{bmatrix}, Z^q = \begin{bmatrix}1\\0\end{bmatrix},$$

so that one may choose $S^q = 0$ and $R^q = 1$, which yields the observer (17) as

$$\dot{\hat{z}}^q = -l_q\hat{z}^q + l_q\tilde{y}, \qquad t \in [(q-1)\tau, q\tau), \quad q\text{: odd},$$

with the initial condition $\hat{z}^q((q-1)\tau) = 0$, and $\tilde{y}$ being the difference between the measured output and the estimated output of (35). The gain l_q will be chosen later by (36). For q even, we take $W^q = I_{2\times 2}$, and $G_q = 0_{2\times 2}$, so that Z^q, S^q, R^q are null-matrices.

The next step is to use the value of $\hat{z}^q(q\tau^-)$ to compute $\xi_q(q\tau^-)$, $q > 3$. We use the notation ξ^q to denote $\xi_q(q\tau^-)$, and let ξ_1^q be the first component of the vector ξ^q. Since $N = 3$, it follows from (14c) that $\xi^1 = \xi^2 = \xi^3 = \mathrm{col}(0, 0)$. The matrices appearing in the computation of ξ^q are given as follows: for every *odd* $q > 3$:

$$\Psi^q_{q-3} = \begin{bmatrix}\cos\tau & \sin\tau\\ -\sin\tau & \cos\tau\end{bmatrix} \Rightarrow \left(\Psi^q_{q-3}I_{2\times 2}\right)^\perp = \{0\},$$

$$\Psi^q_{q-2} = \begin{bmatrix}\cos\tau & \sin\tau\\ -\sin\tau & \cos\tau\end{bmatrix} \Rightarrow \left(\Psi^q_{q-2}\begin{bmatrix}0\\1\end{bmatrix}\right)^\perp = \left\{\begin{bmatrix}\cos\tau\\ -\sin\tau\end{bmatrix}\right\},$$

$$\Psi^q_{q-1} = I_{2\times 2} \Rightarrow \left(\Psi^q_{q-1}I_{2\times 2}\right)^\perp = \{0\},$$

$$\Psi^q_q = I_{2\times 2} \Rightarrow \left(\Psi^q_q\begin{bmatrix}0\\1\end{bmatrix}\right)^\perp = \left\{\begin{bmatrix}1\\0\end{bmatrix}\right\},$$

where the braces $\{\cdot\}$ denote the linear combination of the elements it contains. These subspaces directly lead to the expressions for Θ_j^q, $j = q-3, \ldots, q$, so that

$$\Theta_q = \begin{bmatrix}1 & \cos\tau\\ 0 & -\sin\tau\end{bmatrix}, \quad q = 5, 7, \ldots,$$

where we have used the convention that Θ_i^q is a null matrix whenever $\mathcal{R}(\Psi_i^q(\tau_{\{i+1,q\}})W^i)^\perp = \{0\}$. Hence, the error correction term can be computed recursively for every odd $q > 3$ by the formula:

$$\xi^q = \Theta_q^{-\top}\begin{bmatrix}\hat{z}^q(t_q^-)\\ \hat{z}^{q-2}(t^-_{q-2}) - \xi_1^{q-2} - [\cos\tau \;\; -\sin\tau]\xi^{q-1}\end{bmatrix}.$$

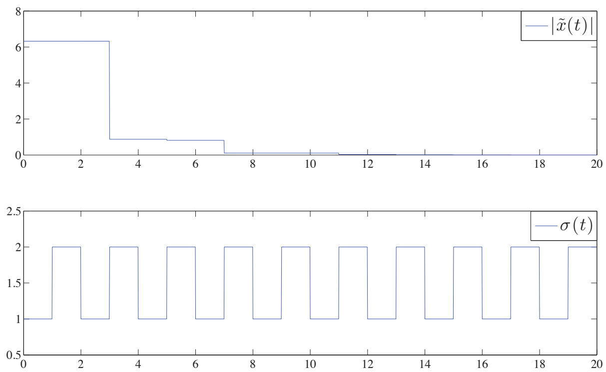

Figure 1: Size of state estimation error and the switching signal

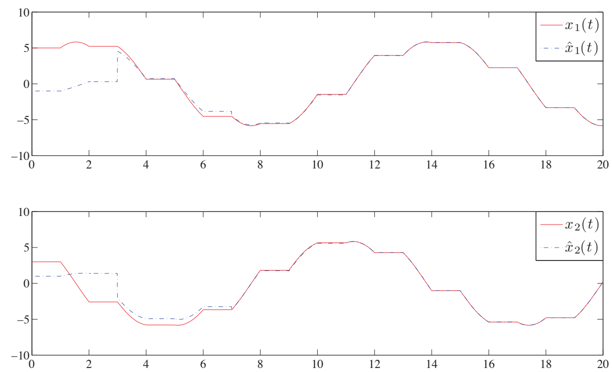

Figure 2: Converging state estimates

Also, for odd q, we obtain that $M_{q-1}^q = 0$, $M_{q-3}^q = 0$,

$$M_q^q = \begin{bmatrix} 1 & 0 \\ \frac{\cos\tau}{\sin\tau} & 0 \end{bmatrix}, \quad \text{and} \quad M_{q-2}^q = \begin{bmatrix} 0 & 0 \\ -\frac{1}{\sin\tau} & 0 \end{bmatrix}.$$

Next, for every *even* $q > 3$, we repeat the same calculations to get:

$$\Psi_{q-3}^q = \begin{bmatrix} \cos 2\tau & \sin 2\tau \\ -\sin 2\tau & \cos 2\tau \end{bmatrix} \Rightarrow \left(\Psi_{q-3}^q \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^{\perp} = \left\{ \begin{bmatrix} \cos 2\tau \\ -\sin 2\tau \end{bmatrix} \right\},$$

$$\Psi_{q-2}^q = \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix} \Rightarrow \left(\Psi_{q-2}^q I_{2\times 2}\right)^{\perp} = \{0\},$$

$$\Psi_{q-1}^q = \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix} \Rightarrow \left(\Psi_{q-1}^q \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^{\perp} = \left\{ \begin{bmatrix} \cos\tau \\ -\sin\tau \end{bmatrix} \right\},$$

$$\Psi_q^q = I_{2\times 2} \Rightarrow \left(\Psi_q^q I_{2\times 2}\right)^{\perp} = \{0\}.$$

Once again, using the expressions for Θ_j^q, $j = q-3, \ldots, q$, based on these subspaces, one gets,

$$\Theta_q = \begin{bmatrix} \cos\tau & \cos 2\tau \\ -\sin\tau & -\sin 2\tau \end{bmatrix}, \quad q = 4, 6, 8, \cdots,$$

so that

$$\xi^q = \Theta_q^{-\top} \begin{bmatrix} \hat{z}^{q-1}(t_{q-1}^-) - \xi_1^{q-1} \\ \hat{z}^{q-3}(t_{q-3}^-) - \xi_1^{q-3} - [\cos\tau \;\; -\sin\tau](\xi^{q-2} + \xi^{q-1}) \end{bmatrix}.$$

Again, we obtain for even q that $M_q^q = M_{q-2}^q = 0$,

$$M_{q-1}^q = \begin{bmatrix} \frac{\sin 2\tau}{\sin\tau} & 0 \\ \frac{\cos 2\tau}{\sin\tau} & 0 \end{bmatrix} \quad \text{and} \quad M_{q-3}^q = \begin{bmatrix} -1 & 0 \\ -\frac{\cos\tau}{\sin\tau} & 0 \end{bmatrix}.$$

By computing the induced 2-norm of a matrix, it is seen that, for any $q > 3$ and $q - 3 \le j \le q$,

$$\|M_j^q Z^j e^{(S_j - l_j R_j)\tau_j} Z^{j\top}\| = \begin{cases} 0 & \text{if } j \text{ is even} \\ \frac{e^{-l_j\tau}}{|\sin\tau|} & \text{if } j \text{ is odd.} \end{cases}$$

Therefore, for the relation (24) and (23), it is enough to choose l_q (for odd q) such that

$$\frac{1}{|\sin\tau|} e^{-l_q\tau} < \frac{1}{N+1} = \frac{1}{4}$$

or,

$$l_q > \frac{1}{\tau} \ln \frac{4}{|\sin\tau|}. \tag{36}$$

Once again it can be seen that the singularity occurs when τ is an integer multiple of π. Moreover, if τ approaches this singularity, then the gain required for convergence gets arbitrarily large. This also explains why the knowledge of the switching signal is required in general to compute the observer gains.

Some results of simulations with $\tau = 1$ and $l_q = 2$ for odd q, are illustrated in Fig. 1 and Fig. 2. The figures clearly show the hybrid nature of the proposed observer, which is caused by the jump discontinuity in error correction. For this particular example, the error does not grow between the switching times because the subsystem 2 is just a rotating dynamics.

4. CONCLUSION

This paper has presented conditions for observability and determinability of switched linear systems with state jumps. Based on these conditions, an asymptotic observer is constructed that combines the partial information obtained from each mode to get an estimate of the state vector. Under the assumption of persistent switching, the error analysis shows that the estimate converges to the actual state. The proposed method relies on the homogeneity that a linear switched system and linear jump maps provide with. In fact, it is seen in (29) that the transportation of the partially observable state information (represented by z), obtained at each mode, can be computed even with some unobservable information (by w). Since homogeneity guarantees that the observable information is not altered by this transportation process, the unobservable components are simply filtered out after the transportation. We emphasize that this idea may not be transparently applied to nonlinear systems.

Acknowledgement

The authors appreciate the discussions with Stephan Trenn related to observability conditions.

Appendix: Proof of Lemma 1

Let $c = \alpha/(N+1)$ with $0 < \alpha < 1$. Then it is obvious that, for $i > N$,

$$|a_i| \le \frac{\alpha}{N+1} \sum_{k=i-N-1}^{i-1} |a_k| \le \alpha \max_{i-N-1 \le k \le i-1} |a_k|. \tag{37}$$

Similarly, it follows that

$$\begin{aligned}|a_{i+1}| &\leq \alpha \max_{i-N\leq k\leq i} |a_k| \\ &\leq \alpha \max\left\{|a_{i-N-1}|, \max_{i-N\leq k\leq i-1} |a_k|, |a_i|\right\} \\ &\leq \alpha \max_{i-N-1\leq k\leq i-1} |a_k|,\end{aligned}$$

where the last inequality follows from (37). By induction, this leads to

$$\max_{i\leq k\leq i+N} |a_k| \leq \alpha \max_{i-N-1\leq k\leq i-1} |a_k|.$$

that is, the maximum value of the sequence $\{a_i\}$ over the length of window $N+1$ is strictly decreasing and converging to zero, which proves the desired result. □

Appendix: Some Useful Facts

Let $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}$ be any linear subspaces, A be a (not necessarily invertible) $n \times n$ matrix, and B, C be matrices of suitable dimension. The following properties can be found in the literature such as [15], or developed with little effort.

1. $A\mathcal{R}(B) = \mathcal{R}(AB)$ and $A^{-1}\ker B = \ker(BA)$.
2. $A^{-1}A\mathcal{V} = \mathcal{V} + \ker A$, and $AA^{-1}\mathcal{V} = \mathcal{V} \cap \mathcal{R}(A)$.
3. $A^{-1}(\mathcal{V}_1 \cap \mathcal{V}_2) = A^{-1}\mathcal{V}_1 \cap A^{-1}\mathcal{V}_2$, and $A(\mathcal{V}_1 \cap \mathcal{V}_2) \subseteq A\mathcal{V}_1 \cap A\mathcal{V}_2$ (with equality if and only if $(\mathcal{V}_1 + \mathcal{V}_2) \cap \ker A = \mathcal{V}_1 \cap \ker A + \mathcal{V}_2 \cap \ker A$, which holds, in particular, for any invertible A).
4. $A\mathcal{V}_1 + A\mathcal{V}_2 = A(\mathcal{V}_1 + \mathcal{V}_2)$, and $A^{-1}\mathcal{V}_1 + A^{-1}\mathcal{V}_2 \subseteq A^{-1}(\mathcal{V}_1 + \mathcal{V}_2)$ (with equality if and only if $(\mathcal{V}_1 + \mathcal{V}_2) \cap \mathcal{R}(A) = \mathcal{V}_1 \cap \mathcal{R}(A) + \mathcal{V}_2 \cap \mathcal{R}(A)$, which holds, in particular, for any invertible A).
5. $(\ker A)^\perp = \mathcal{R}(A^\top)$.
6. $(A^\top\mathcal{V})^\perp = A^{-1}\mathcal{V}^\perp$ and $(A^{-1}\mathcal{V})^\perp = A^\top\mathcal{V}^\perp$.
7. $\langle A|\mathcal{V}\rangle = \mathcal{V} + A\mathcal{V} + A^2\mathcal{V} + \cdots + A^{n-1}\mathcal{V}$ and $\langle \mathcal{V}|A\rangle = \mathcal{V} \cap A^{-1}\mathcal{V} \cap A^{-2}\mathcal{V} \cap \cdots \cap A^{-(n-1)}\mathcal{V}$.
8. $\langle \mathcal{V}_1 \cap \mathcal{V}_2|A\rangle = \langle \mathcal{V}_1|A\rangle \cap \langle \mathcal{V}_2|A\rangle$ and $\langle A|\mathcal{V}_1 \cap \mathcal{V}_2\rangle \subset \langle A|\mathcal{V}_1\rangle \cap \langle A|\mathcal{V}_2\rangle$.
9. $e^{At}\mathcal{V} \subseteq \langle A|\mathcal{V}\rangle$ and $\langle \mathcal{V}|A\rangle \subseteq e^{At}\mathcal{V}$ for any t.
10. $\langle A|\mathcal{V}\rangle^\perp = \langle \mathcal{V}^\perp|A^\top\rangle$.

Now, with $G := \mathrm{col}(C, CA, \ldots, CA^{n-1})$,

11. $e^{At}\ker G = \ker G$ and $e^{A^\top t}\mathcal{R}(G^\top) = \mathcal{R}(G^\top)$ for all t.
12. $\langle \ker G|A\rangle = \ker G$ and $\langle A^\top|\mathcal{R}(G^\top)\rangle = \mathcal{R}(G^\top)$.

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