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Using AVK method to solve nonlinear problems with uncertain parameters

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Abstract

Although we usually would like to work with exact systems, most of the real world systems are nonlinear with uncertain parameters. In this paper, we propose AVK (A.V. Kamyad) approach to solve nonlinear problems with uncertain parameters (NPUP). This approach substitutes the original nonlinear system with an equivalent nonlinear programming (NLP) problem. Using this approach, we found an optimal solution of NLP problem and a new approximate solution for the original NPUP in L_p space.

Keywords: Uncertain parameter; Nonlinear problem; Optimal control; AVK method

1. Introduction

Uncertainty parameters usually appear in systems with undefined or unknown parameters. The solution of these systems often is difficult and achieving the control of the system is boring. In recent years, many approaches were proposed for analysis of uncertainty such as robust control feedback, Gain Loci characteristic, μ synthesis, LQR/LTR, H^2 and H^∞ [2,5], adaptive control, Gain scheduling, Genetic Algorithm, neural network, fuzzy control might be used for control of uncertain parameter systems [6,7]. Many of them work with any certain class of nonlinear system or linear systems.

Many authors try to consider certain parameters or use stochastic or probability structure with known mean value and variance of probability distribution (Bayesian Models) [1]. Also we may assume parameters unknown but bounded (Fisher Models) [2]. Uncertainties approaches are shown in two different categories; structured uncertainties (Model uncertainty) and unstructured uncertainties (Data uncertainty) [3]. Uncertainty appears in systems with a different variety, for example there are additive ($G = \bar{G} + \Delta_A$) or multiplicative uncertainties ($G = (1 + \Delta_M)\bar{G}$), where G is real model of system, $\bar{G}$ is nominal model, Δ_A and Δ_M are additive and multiplicative uncertainty, respectively [4].

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For the first time, A.V. Kamyad (AVK) et al. proposed an approach that may find optimal control for NPUP [8,9] to solve an equivalent problem by a linear programming. In this paper we extend former method and propose new approach to solve NPUP by NLP. The algorithm introduces an approximate solution of the NPUP based on optimization [10–17]. In this approach we define an equivalent minimization problem for the NPUP and substitute this system with an equivalent NLP problem by discretizing. In fact, this is the approximate solution of the original problem which is the best solution for the original NPUP in L_p space. Moreover the error of this approximated solution is controllable. Finally our algorithm will be confirmed by simulation of different NPUP.

The following notation is used in this paper. $x \in R^n$ denotes an element of the n-dimensional Euclidian space with arbitrary norm function $\|X\|_{L_1}$ where is defined as follows:

$$\|X\|_{L_1} = \|(x_1(t), x_2(t), \ldots, x_n(t))\|_{L_1} = |x_1(t)| + |x_2(t)| + \cdots + |x_n(t)|.$$

2. Definitions and AVK method

2.1. Nonlinear problems with uncertain parameter (NPUP)

Definition 1. We focus on following NPUP:

$$\begin{aligned} &\dot{x} = f(x, u, \Delta) \\ &\text{s.t. } x(a) = x_a, \quad x(b) = x_b, \end{aligned} \tag{1}$$

where f is a continuous nonlinear time varying function from $X \times U \times [a, b]$ to R^n, so that $x(t) \in X \subset R^n$ is state function, $u(t) \in U \subset R^m$ is control function and $t \in [a, b] \subseteq R$ is independent variable so called, time. X and U are compact subsets and must be chosen as the system reaches from initial state $x(a)$ to final state $x(b)$. We suppose the nonlinear system (1) is stable in $[a, b]$ interval. Also Δ is system uncertainty and it is generally unknown but bounded i.e. $a_1 \leqslant \Delta \leqslant a_2$ and $a_1, a_2 \geqslant 0$. The pair $x^*(t)$ and $u^*(t)$ are the solution of problem (1) if they track the desired curve $x_d(t)$. For full tracking, the initial state must be as followed: $x_d(0) = x(a)$ and x_a, x_b are initial and final state in R^n respectively, that may be fixed or free.

Definition 2. First, consider nonlinear system (1), we define following functional that is called the total error functional. Let

$$E_1(x(t), u(t), \Delta) = \int_a^b \|\dot{x} - f(x, u, \Delta)\| \, \mathrm{d}t, \tag{2}$$

where $E_1 : X \times U \times [a, b] \to R$ is a continuous functional.

Definition 3. The solution of uncertainty problem can track the desired curve $x_d(t)$, if we consider a multi objective functional as

$$E_2(x, u, \Delta) = \int_a^b (\|x(t) - x_d(t)\| + \|\dot{x} - f(x, u, \Delta)\|) \, \mathrm{d}t. \tag{3}$$

Definition 4. If Δ has a known distributed function similar to $g(t)$, we may define new functional:

$$E_3(x, u, \Delta) = \int_a^b (\|x(t) - x_d(t)\| + g(t)\|\dot{x} - f(x, u, \Delta)\|) \, \mathrm{d}t. \tag{4}$$

Now, the following key theorem is demonstrated.

Theorem 1. *If h is a nonlinear continuous function on $X \times U \times [a, b]$ and non-negative ($h \geqslant 0$), then the necessary and sufficient condition for $\int_a^b h \, \mathrm{d}x = 0$ is $h \equiv 0$ on $[a, b]$.*

Proof. Let assume $\int_a^b h\,\mathrm{d}x = 0$ but $h \neq 0$ and by assumption at a point x in $[a,b]$, $h(x) > 0$, since for continuity of $h(x)$ it is positive in some neighborhood of x i.e. $h(x) > 0$ for all $x \in (x_1 - \varepsilon, x_1 + \varepsilon)$ that ε is a positive number. Therefore $\int_a^b h\,\mathrm{d}x \geqslant \int_{x_1-\varepsilon}^{x_1+\varepsilon} h\,\mathrm{d}x > 0$ i.e. $\int_a^b h\,\mathrm{d}x > 0$, a contradiction to our assumption. Thus h must be zero on $[a,b]$. On the other hand if $h \equiv 0$ on $X \times U \times [a,b]$ then obviously $\int_a^b h\,\mathrm{d}x = 0$. □

Theorem 2. *Necessary and sufficient condition for a nonlinear function to be concluded a NPUP (1), with initial condition $x(a)$ and final state $x(b)$ is to satisfy the following relation in problem (2):*

$$E_1(x,u,\Delta) = 0.$$

Proof. It is sufficient to define h in Theorem 1 as follows:

$$h(t) = \|\dot{x} - f(x,u,\Delta)\|_{L_1}. \tag{5}$$

Since $\|\cdot\|_{L_1}$ is a continuous function and non-negative and also $f(x,u,\Delta)$ is continuous function, then $h(t)$ is continuous with respect to variables $x, u, \Delta, \dot{x}$ and since $x, u, \Delta, \dot{x}$ are continuous functions on $[a,b]$, then total function $\dot{x} - f(x,u,\Delta)$ is continuous in interval $[a,b]$. Therefore, using Theorem 1 $\int_a^b h(t)\,\mathrm{d}t = 0$ is equivalent to $h(t) \equiv 0$ for all $t \in [a,b]$ i.e.

$$\dot{x} = f(x,u,\Delta) \quad \forall t \in [a,b]. \tag{6}$$

Therefore, Theorem 2 is proved. □

Note: Without loss of generality, we may assume $a = 0$ and $b = 1$ thus interval (a,b) is converted to $(0,1)$.

Note: We can assume $h(t)$ is a non-negative piecewise continuous functional in $[0,1]$ instead of non-negative continuous condition in $[0,1]$ and for $h = 0$ we may assume $h(t) = 0$, almost everywhere in $[0,1]$.

2.2. AVK method

In AVK method, the following problem is defined in calculus of variations:

$$\begin{aligned} &\underset{x,u,\Delta}{\text{Minimize}} \quad E_3(x,u,\Delta) = \int_0^1 (|x(t) - x_d(t)| + g(t)|\dot{x} - f(x,u,\Delta)|)\,\mathrm{d}t \\ &\text{s.t.} \qquad\qquad x(0) = x_a, \quad x(1) = x_b, \end{aligned} \tag{7}$$

where $(x,u,t) \in X \times U \times [0,1]$. For all uncertain values, $a_1 \leqslant \Delta \leqslant a_2$, $g(t)$ is a distribution function. We assume the optimal solution of problem (7) is $x^*(t), u^*(t)$, the state and the control functions, respectively. And according to Theorems 1 and 2:

$$E_3(x^*, u^*, \Delta) = 0, \tag{8}$$

i.e.

$$\begin{aligned} &\dot{x}^* = f(x^*, u^*, \Delta), \\ &x^*(t) \cong x_d(t), \quad t \in [0,1], \\ &x^*(0) = x_d(0) = x_a, \quad x^*(1) = x_d(1) = x_b. \end{aligned} \tag{9}$$

Then in general, for solving NPUP (1) we can solve the minimization problem (7) by Theorems 1 and 2. Thus the optimal solution of problem (1) is $x^*(t), u^*(t)$ from optimization problem (7).

2.3. Using AVK method to optimize NPUP

2.3.1. Non-consistency

We define non-consistency problem as an example. Let consider the following system of equations:

$$\begin{cases} x_1 + x_2 = 1, \\ x_1 + x_2 = 2, \\ x_1 + x_2 = 3. \end{cases} \tag{10}$$

This system is called non-consistent. For solving the system, total error must be minimized. We define total error as

$$E_{\text{total}} = |x_1 + x_2 - 1| + |x_1 + x_2 - 2| + |x_1 + x_2 - 3|.$$

And we solve the following NLP problem:

$$\text{Minimize} \quad E_{\text{total}} = |x_1 + x_2 - 1| + |x_1 + x_2 - 2| + |x_1 + x_2 - 3|, \tag{11}$$

where x_1, x_2 are free and real numbers. If x_1, x_2 are the optimal solution of (11) we may say that x_1^*, x_2^* are the best suggested solution of the system (10).

2.3.2. Optimization of NPUP

We have $a_1 \leqslant \Delta \leqslant a_2$ in equivalent problem (7) so that Δ takes all values between a_1, a_2. Then divide the interval $[a_1, a_2]$ is divided by n equal part. Let $\alpha_k = k\frac{a_2 - a_1}{n}$

$$\Delta_k = a_1 + \alpha_k,$$

where $k = 0, 1, \ldots, n$, $\Delta_0 = a_1$ and $\Delta_n = a_2$. We define non-consistent differential equation system similar to the equation system (10):

$$\begin{cases} \dot{x} = f(x, u, \Delta_0) = f(x, u, a_1), \\ \dot{x} = f(x, u, \Delta_1), \\ \vdots \\ \dot{x} = f(x, u, \Delta_n) = f(x, u, a_2). \end{cases} \tag{12}$$

We are looking for the best solution $x^*(t), u^*(t)$ for the non-consistent differential equation system (12) where all discrete values of Δ are included. The best solution for the optimization problem (7) is minimizing the total error of above system, i.e. total error in L_1 space is minimized as follows:

$$\begin{aligned} &\underset{x,u}{\text{Minimize}} \quad \int_0^1 \{|x(t) - x_d(t)| + g(t)(|\dot{x} - f(x, u, \Delta_0)| + \cdots + |\dot{x} - f(x, u, \Delta_n)|)\}\,\mathrm{d}t \\ &= \int_0^1 \left(|x(t) - x_d(t)| + g(t)\sum_{k=0}^{n} |\dot{x} - f(x, u, \Delta_k)|\right)\mathrm{d}t, \end{aligned} \tag{13}$$

$$x_d(0) = x_a, \quad x_d(1) = x_b.$$

Now, we will solve problem (13) approximately.

2.4. Discretization

We partition the interval $t \in [0, 1]$ to m equal subintervals (cells), where m is arbitrary fixed positive integer, then problem (13) yields to

$$\underset{x,u}{\text{Minimize}} \quad \sum_{i=1}^{m} \int_{(i-1)/m}^{i/m} \left(|x(t) - x_d(t)| + g(t)\sum_{k=0}^{n} |\dot{x} - f(x, u, \Delta_k)|\right)\mathrm{d}t.$$

For summarize, the initial value $x_d(0) = x_a$ and final value $x_d(1) = x_b$ are neglected. Let $\delta t = \frac{1}{m}$, for the first derivative we have

$$\dot{x}(t) \cong \frac{x(t + \delta t) - x(t)}{\delta t}.$$

Suppose $\delta t \to 0$, thus the approximate value achieves to the best value for derivation at the time t. Hence δt or sampling time is very important, and must be chosen small, so the number of partitions is great. This is a trade off between sampling time and speed of problem solving. Also, we use L_1 norm as follows:

$$\underset{x,u}{\text{Minimize}} \quad \sum_{i=1}^{m} \int_{(i-1)/m}^{i/m} \left(|x(t) - x_d(t)| + g(t)\sum_{k=0}^{n} \left|m\left[x\left(t + \frac{1}{m}\right) - x(t)\right] - f(x, u, \Delta_k)\right|\right)\mathrm{d}t.$$

Remark. As we know, an approximate value of integral $\int_a^b K(x)\,dx$ is $(b-a)K(c)$, where c is any point such as $a \leqslant c \leqslant b$.

So, applying above remark, and assume c is an ending point in any subinterval, minimization problem (13) is formed as

$$\underset{x,u}{\text{Minimize}} \quad \sum_{i=1}^{m} \frac{1}{m}\left(\left|x\left(\frac{i}{m}\right)-x_d\left(\frac{i}{m}\right)\right| + g\left(\frac{i}{m}\right)\sum_{k=0}^{n}\left|m\left[x\left(\frac{i+1}{m}\right)-x\left(\frac{i}{m}\right)\right]-f\left(x\left(\frac{i}{m}\right),u\left(\frac{i}{m}\right),\Delta_k\right)\right|\right)$$
$$=\sum_{i=1}^{m}\sum_{k=0}^{n}\frac{1}{m}\left(\left|x\left(\frac{i}{m}\right)-x_d\left(\frac{i}{m}\right)\right| + g\left(\frac{i}{m}\right)\left|m\left[x\left(\frac{i+1}{m}\right)-x\left(\frac{i}{m}\right)\right]-f\left(x\left(\frac{i}{m}\right),u\left(\frac{i}{m}\right),\Delta_k\right)\right|\right). \tag{14}$$

So define the unknown parameters as

$$t_i = \tfrac{i}{m}, \qquad x_i = x\left(\tfrac{i}{m}\right), \quad i=1,2,\ldots,m, \qquad x_{d_i} = x_d\left(\tfrac{i}{m}\right).$$

Thus $t_0 = 0$, $t_m = 1$, $x_0 = x_{d_0} = x_a$, $x_m = x_{d_m} = x_b$, and $x_{i+1} = x(\frac{i}{m}+1)$. Also assume

$$g_i = g\left(\tfrac{i}{m}\right), \quad u_i = u\left(\tfrac{i}{m}\right). \quad i=1,2,\ldots,m, \tag{15}$$

Thus, we simplify obtained discretized problem (14) in the form

$$\begin{array}{ll}\underset{x_i,u_i}{\text{Minimize}} & \sum_{i=1}^{m}\sum_{k=0}^{n}\frac{1}{m}[|x_i - x_{d_i}| + g_i|m[x_{i+1}-x_i] - f(x_i,u_i,\Delta_k)|] \\ \text{s.t.} & x_0 = x_a, \quad x_m = x_b.\end{array} \tag{16}$$

As a whole, problem (16) is a NLP problem and we may obtain its solution by many packages such as Lingo, Matlab, Gino, etc. Finally, by obtaining the solution of problem (16), we recognize the value of unknown admissible pair (x_i, u_i) state and control function at $n \times l$ points. We can construct the optimal solution for NPUP (1) by two piecewise functions (x_i^*, u_i^*). Theorem 2 will shows existence of the optimal solution for the NPUP (1).

3. Simulation

In this section, we use our algorithm for some NPUP

Example 1. Assume second order $\ddot{x} + a(t)\dot{x}\cos 3x = u$ where $a(t)$ is unknown but bounded $1 \leqslant a(t) \leqslant 2$ and desired trajectory is $x_d(t) = \sin(2\pi t)$ [18], controller is bounded $u(t) \in [-1, .8]$.

Solution. For this system we choose partitioning [0,1] to 10 equal subintervals. So $t_0 = 0, t_1 = .1, \ldots, t_{10} = 1$. The formulation of corresponding NLP problem is as the following:

$$\begin{array}{ll}\text{Minimize} & J(\cdot,\cdot) = \int_0^1 (|\sin(2\pi t) - x(t)| + |\ddot{x} + a(t)\dot{x}\cos 3x - u|)\,dt \\ \text{s.t.} & x(0) = 0, \quad x(1) = 1 \quad \text{and} \quad u(t) \in [-1, .8].\end{array}$$

The trajectories of approximate state function (bold line) with desired function (dotted line) are shown in Fig. 1 and control function is schemed in Fig. 2.

And with unbounded controller we have better trajectory tracking, which is shown in Figs. 3 and 4.

Example 2. Assume Example 1 so that Δ has normal probability structure i.e. $\Delta \sim N(\mu, \sigma)$. Present a piecewise continuous controller.

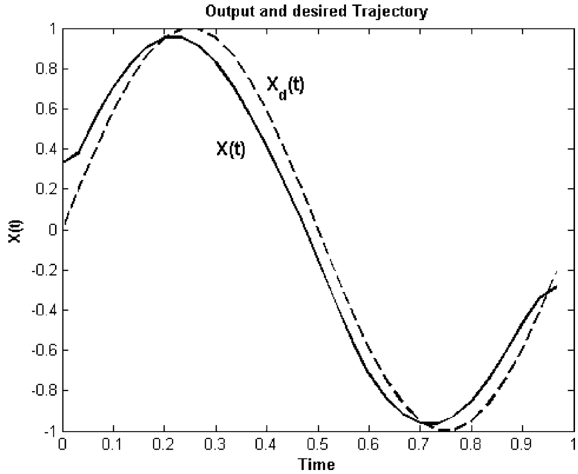

Fig. 1. Approximate solution of the state function (bold line) and desired function (dotted line).

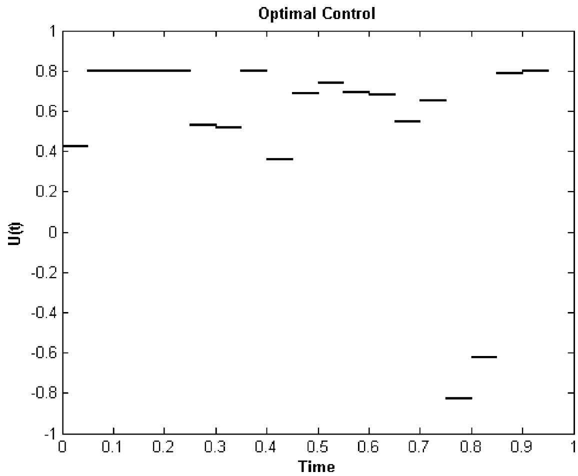

Fig. 2. The control function.

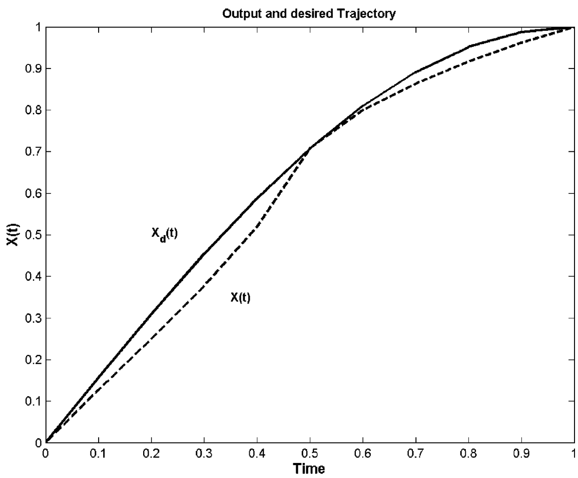

Fig. 3. Approximate solution of the state function (bold line) and desired function (dotted line).

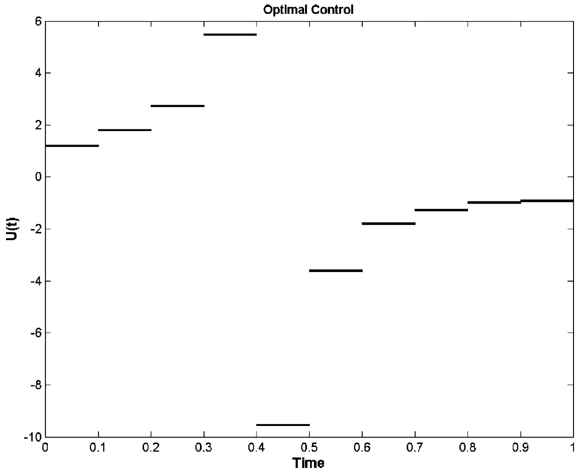

Fig. 4. The control function.

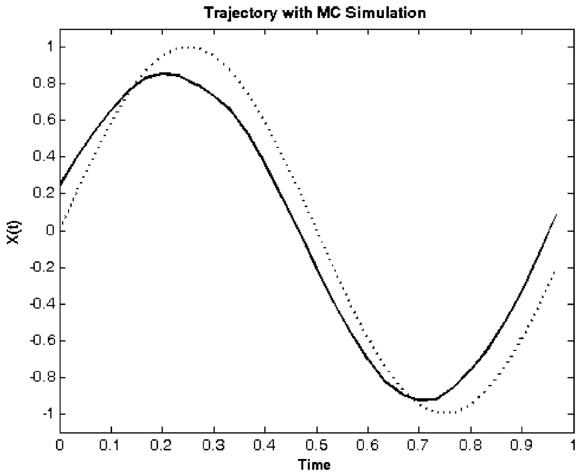

Fig. 5. Fifty times MC simulation for approximate state (bold line) and desired function (dotted line).

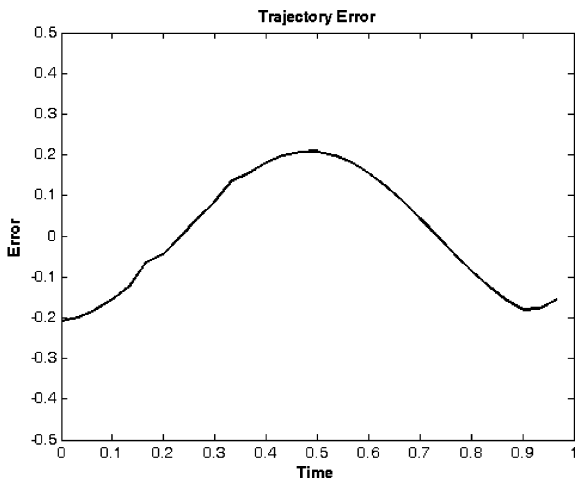

Fig. 6. Trajectory error.

Solution. We may write $g(t)$ in minimization problem (7) as $g(t) = \mathrm{e}^{-\left(\frac{t-\mu}{\sigma}\right)^2}$, where $\mu = 1.5$ and $\sigma = 1$. Then the trajectories of approximate state function (bold line) and desired function (dotted line) with 50 times Monte Carlo (MC) simulation are presented in Fig. 5 and its error in Fig. 6.

4. Conclusion

Operating uncertainty in nonlinear systems and discretizing an NPUP to a NLP to obtain an approximate solution of the original problem is the main goal of this paper. In this article, we analyze uncertainty and nonlinearity in a general form. Our approach introduces an approximate solution for the NPUP based on optimization. Then the problem is transferred to a new problem in form of variations calculus. By discretizing the new problem and solving it by using NLP packages, we obtained the best approximate solution of the original NPUP. Simulations confirmed the efficacy of our approach in tracking desired trajectory and solving NPUP in comparison with the result obtained in [8]. Moreover, comparing with [18], the AVK algorithm yields a more practical piecewise continuous controller.

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