A NEW $\mathscr{H}_\infty$ DESIGN METHOD FOR HIGH PERFORMANCE ROBUST TRACKING CONTROL

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Abstract: In many control design problems it is usually of great interest to have direct control over the closed-loop bandwidth and the transfer function from reference to plant output. The Internal Model Control (IMC) design method for stable plants offers this feature. However, there are some shortcomings, inadequacies and limitations with the IMC design method for both stable and unstable plants which will be exposed in this paper. With the aim of keeping the desired features of the IMC design method, we propose a new $\mathscr{H}_\infty$ control design method which addresses open issues and also can be used for both stable or unstable plants in a coherent framework.

Keywords: *$\mathscr{H}_\infty$ Control, Internal Model Control, Model Matching, Robust Tracking, Adaptive Control*

1. INTRODUCTION

In this paper, we propose a new controller design method that preserves the desirable features and much of the simplicity of the Internal Model Control (IMC) design method (Morari and Zafiriou, 1989) but extends its effective range of application. In particular, we shall capitalize on the exceptional feature of the IMC design method which in principle allows—for stable plants with no $j\omega$-axis zeros—the design of a controller that achieves a closed-loop magnitude response exactly equal to that of a desired transfer function. In the IMC framework, this choice of transfer function is known as the IMC filter, as explained in Section 2, and it permits inclusion of a single parameter which tunes the bandwidth of the designed closed-loop system. This important bonus gives IMC extra appeal in the area of adaptive robust control where it is desired to have direct control over the closed-loop bandwidth and hence the ability to progressively increase the bandwidth in identification and controller re-design. (Lee *et al.*, 1995; Anderson, 2002).

For *unstable* plants, however, the IMC scheme is much more involved, to say the least. For example, the above-mentioned single design parameter does not directly tune the closed-loop bandwidth (Morari and Zafiriou, 1989; Campi *et al.*, 1982). Research directed at finding solutions for the above-mentioned problems have resulted in design methods which are nonetheless application-specific (e.g. excluding unstable plants with unstable zeros, and requiring additional parameter tuning to trade-off the magnitude of the overshoot and the settling time in the step response) (Campi *et al.*, 1982; Lee *et al.*, 2001).

Even for *stable* plants, the IMC design method may result in an unacceptable design, e.g. pole-zero cancellation may occur very close to the $j\omega$-axis, and the design method cannot handle plants with $j\omega$-axis zeros.

The IMC design method also deals specifically with the transfer function from input r_1 to output y in Fig. 1, which is the complementary sensitivity $T_{yr_1} = PC/(1 + PC)$, by setting its magnitude response (see Section 2), but it does not explicitly handle the size of other transfer functions (T_{yr_2},T_{ur_1},T_{ur_2}). However, the other three transfer functions do relate to certain input-output properties of a feedback loop as discussed

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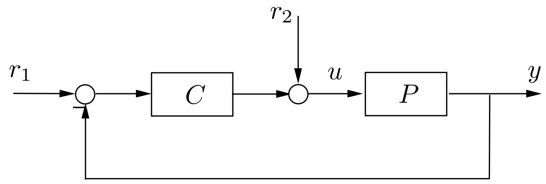

Fig. 1. Standard Feedback Configuration

in Section 2, and the IMC design method may fail to ensure that their values are acceptable.

We shall introduce an $\mathscr{H}_\infty$ controller design method with the prime intention of maintaining the desirable feature of the IMC discussed above (including control over the closed-loop bandwidth via adjustment of a single parameter), but also addressing the potential difficulties with the IMC design method (Section 3). This proposed controller design method can be used for both stable and unstable plants and extends the domain of applicability by addressing some of the IMC limitations, which are pointed out in Section 2 and addressed in Section 3. The proposed controller design method of Section 4 relies on an $\mathscr{H}_\infty$ control problem that can be easily solved using standard software. The versatility of the proposed $\mathscr{H}_\infty$ design technique is illustrated in Section 5.

2. A CRITIQUE OF IMC

We shall outline the IMC design method for both stable and unstable plants in order to highlight further the difficulties and the circumstances where the IMC design method cannot be properly used, or its applicability is limited by restrictive assumptions, or it fails to provide a good design.

Consider the unity feedback system in Fig. 1 with the closed-loop mapping given by

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \frac{PC}{1+PC} & \frac{P}{1+PC} \\ \frac{C}{1+PC} & \frac{1}{1+PC} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = H(P,C) \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \quad (1)$$

for which $H(P,C) \in \mathscr{RH}_\infty$ satisfies internal stability requirements (Zhou *et al.*, 1996). We shall explore this relationship and the requirements of internal stability for the IMC design method in the sequel.

Following (Morari and Zafiriou, 1989), stable plants, which are simpler, are treated separately from unstable plants. Recall the feedback system in Fig. 1 and suppose $P \in \mathscr{RH}_\infty$ with no $j\omega$-axis zero[2]. Decompose P multiplicatively into a stable all-pass P_a, and a stable minimum-phase P_m component, that is

$$P = P_a\, P_m\ . \quad (2)$$

The design goal is approached by choosing an "IMC filter transfer function" $F(s)$ such that with an appropriate controller, the closed-loop transfer function is

$$T_{yr} = \frac{PC}{1+PC} = P_a F \quad (3)$$

thus $|T_{yr_1}| = |F|$. A common choice for F is

$$F = \left(\frac{\lambda}{s+\lambda}\right)^n \quad (4)$$

for some λ which specifies the bandwidth of T_{yr_1}, and some positive integer n which should be at least equal to the relative degree of the plant. The so-called Q-parameter defining the controller

$$C = \frac{Q}{1-PQ} \quad (5)$$

which achieves (3) is easily found to be $Q = P_m^{-1}F$ and $|T_{yr_1}| = |F|$.

Evidently $Q \in \mathscr{RH}_\infty$, provided that $F \in \mathscr{RH}_\infty$ and the relative degree of F is at least equal to that of P_m, guarantees internal stability with the parameterization given in (5).

Note that in terms of Youla-Kucera parameterization (Youla *et al.*, 1976; Kucera, 1979) ideas, Equation (5) is a formula for all stabilizing controllers given stability of P. Let $P = NM^{-1}$, with N and M right coprime over $\mathscr{RH}_\infty$, and let $C_0 = UV^{-1}$, with U and V right coprime over $\mathscr{RH}_\infty$, be a controller that internally stabilizes the feedback system in Fig. 1. Then all controllers for which the feedback system in Fig. 1 is internally stable are given by $C = (U + M\hat{Q})/(V - N\hat{Q})$ for any $\hat{Q} \in \mathscr{RH}_\infty$. Then for $P \in \mathscr{RH}_\infty$, choosing $N = P$, $M = 1$, $U = 0$ and $V = 1$ will result in the parameterization given in Equation (5).

The above discussions confirm the simplicity and efficiency of the design method for stable plants. Nevertheless, the method has limitations, depends on restrictive assumptions and gives rise to certain open problems that we will explain below.

The IMC design method for unstable plants, however, requires substantial adjustment (Morari and Zafiriou, 1989). Let us again seek a controller parameterization for the design of C as in Equation (5) and develop the requirements on C that ensure internal stability. Since

$$H(P,C) = \begin{bmatrix} PQ & (1-PQ)P \\ Q & 1-PQ \end{bmatrix} \in \mathscr{RH}_\infty \quad (6)$$

it is necessary and sufficient that following conditions be satisfied: i. $Q \in \mathscr{RH}_\infty$ as before and ii. $(1 - PQ) = 0$ at the closed right half-plane poles of P (Morari and Zafiriou, 1989, Thm. 5.1-1). Hence, the parameterization in Equation (5) ought to be evolved to meet these conditions on Q. All Q for which $1 - PQ = 0$ at the closed right half-plane poles of P are given in (Morari and Zafiriou, 1989, Thm. 5.1-2) through the introduction of a two-step procedure for parameterization of stabilizing controllers which is indeed an alternative to the standard Youla parameterization discussed above.

However, notice that any choice of $F \in \mathscr{RH}_\infty$ with relative degree of at least that of P, with the closed right half-plane poles a_i, *and additionally such that*

$$[P_a(s)\, F(s)]_{s=a_i} = 1 \quad (7)$$

[2] Note that the decomposition in Equation (2) cannot be achieved if P has a $j\omega$-axis zero.

will result, after retaining the choice $Q = P_m^{-1}F$, in a stabilizing controller achieving $T_{yr_1} = P_a F$. Evidently, we cannot expect that the choice F as in (4) will meet the requirement in (7). A different transfer function for the filter F is given which will be discussed in Section 3.1.

One might assert that using the standard Youla-Kucera parameterization (Youla *et al.*, 1976; Kucera, 1979) will facilitate the design since there would be no need to satisfy the different condition ii. above on Q. It is not hard to verify that using the standard Youla-Kucera controller parameterization discussed above will result in all four transfer functions in $H(P, C)$ being affine in $\hat{Q}$. However, attaining a desired amplitude response for the closed-loop transfer function T_{yr_1} with the proper choice of $\hat{Q}$ (as can be done in the stable case) proves to be harder. This will be clear in Section 3.1 when we discuss the fundamental limitation on achievable closed-loop performance when the plant is unstable (Freudenberg and Looze, 1985). Abstractly though, it is clear that working with a constrained Q or unconstrained $\hat{Q}$ is equivalent.

Let us now revisit Equation (1) to elaborate on the importance of each entry of $H(P, C)$ and also motivate the importance of considering the four transfer functions. The (1,1) entry of $H(P, C)$, which is the complementary sensitivity $T_{yr_1} = PC/(1 + PC)$, is clearly important for reference tracking. The IMC design method sets its magnitude response (see Equation (3)) but only ensures that the other transfer functions in $H(P, C)$ are stable *but does not explicitly handle their size.* The reciprocal of the size of $\|H(P, C)\|_\infty$ is referred to as the generalized robust stability margin (Vinnicombe, 2000) and it corresponds to the amount of (coprime factor) uncertainty that can perturb P without destabilizing the loop (Zhou *et al.*, 1996). Thus, we clearly wish $\|H(P, C)\|_\infty$ to be small for a robust design in this sense. With this introduction in mind, let us now record six different circumstances where the IMC design method discussed above cannot be properly used, or its applicability is limited by restrictive assumptions, or it fails to provide a good design.

First, if P has lightly-damped stable poles in the closed-loop passband, then $P/(1 + PC)$ will have large gain near the frequencies of those poles. Second, if P has lightly-damped stable or unstable zeros in the closed-loop passband, then $C/(1 + PC)$ will have large gain near the frequencies of those zeros. This will have direct impact on the maximum singular value of $H(P, C)$, $\bar{\sigma}[H(P, C)]$, as large T_{yr_2} or large T_{ur_1} at some frequency means large $\bar{\sigma}[H(P, C)]$ and hence poor design. Third, if the bandwidth of F is chosen to be much larger than that of P, then $|C|$ will be very large at frequencies inside the bandwidth of F and outside the bandwidth of P which will result in $\bar{\sigma}[H(P, C)]$ being large and hence poor design.

Fourth, the controller C becomes improper if the roll-off rate of F is desired to be less than that of P. Fifth, the simple decomposition in Equation (2) is not possible if P has zeros on the $j\omega$-axis[3]. Sixth, unstable plants pose problems.

In the following section, we shall introduce a new controller design method that inherits the useful desired features of the IMC design method, but explicitly addresses the above-stated problems.

3. THE PROPOSED $\mathscr{H}_\infty$ CONTROL DESIGN METHOD

We wish to accomplish two primarily performance objectives:

- to have $\frac{PC}{1+PC}$ close to *but not necessarily equal to* a target $P_a F$ even with P unstable or perhaps possessing a $j\omega$-axis zero;
- to make sure that the other three transfer functions in (1) do not take large magnitudes.

Obviously, these two objectives are not the same and we shall introduce an $\mathscr{H}_\infty$ index and require this index to be minimized over all stabilizing controllers. Moreover, we normally have performance objectives in mind, which often require some transfer functions to be small or below certain values in some frequency regions and other transfer functions to be small or below certain values at other frequencies. The $\mathscr{H}_\infty$ index will be weighted to achieve the desired effect. Obviously, there will be a trade-off between keeping the size of $\left[\frac{PC}{1+PC} - P_a F\right]$ small and the size of the other three transfer functions in (1) below certain values. We shall now briefly sketch our proposed controller design method here and present a step-by-step procedure in Section 4.

i. Given a model of the plant P do the following factorization:

$$P = P_a P_m, \text{where } \begin{cases} P_a \in \mathscr{RH}_\infty ,\ P_a^{\sim} P_a = I \\ P_m \ \ \textit{has no zeros in}\ \mathbb{C}_+ \end{cases} \qquad (8)$$

where $\mathbb{C}_+$ denotes the open right-half plane.

ii. The admissible controller is given by solving the following $\mathscr{H}_\infty$ problem:

$$\gamma = \min_{C \in \underline{\mathcal{C}}} \left\| \begin{matrix} \frac{PC}{1+PC} - P_a F & \varepsilon_2(s)\frac{P}{1+PC} \\ \varepsilon_1(s)\frac{C}{1+PC} & \varepsilon_1(s)\varepsilon_2(s)\frac{1}{1+PC} \end{matrix} \right\|_\infty \qquad (9)$$

where $\underline{\mathcal{C}}$ denotes the set of all proper stabilizing controllers for the plant P and $\varepsilon_1(s)$ and $\varepsilon_2(s)$ are SISO, stable, minimum-phase and proper weights[4].

We shall explain in detail the selection of F and the weighting functions $\varepsilon_1(s)$ and $\varepsilon_2(s)$ in the following subsections.

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[3] Note that the IMC design procedure assumes that P does not have any zeros on the $j\omega$-axis.

[4] Note that $\min_{C \in \underline{\mathcal{C}}} \|\mathcal{F}_l(.,.)\|_\infty$ rather than $\inf_{C \in \underline{\mathcal{C}}} \|\mathcal{F}_l(.,.)\|_\infty$ is used since we need both γ and $C \in \underline{\mathcal{C}}$. That is, the controller C must be proper and stabilizing and must achieve γ. In the cases we consider, the minimum is attained.

Note that the proposed design method outlined above addresses the problems discussed in Section 2 concerning the difficulties with the IMC design method. This design method is applicable to stable or unstable plants, plants with or without $j\omega$-axis zeros or lightly-damped poles/zeros, and the filter $F(j\omega)$ can have a roll-off rate larger or smaller than that of $P(j\omega)$, and a bandwidth that is larger or smaller than that of $P(j\omega)$. Furthermore, the set $\underline{\mathcal{C}}$ contains stabilizing proper controllers and hence internal stability will always be achieved and the controller will always be proper.

One can easily verify that the *assumptions of a standard $\mathscr{H}_\infty$ control problem* for the index in (9) are fulfilled when $\varepsilon_1(s)$ is chosen to be bi-proper. The reader is referred to (Zhou *et al.*, 1996; Green and Limebeer, 1995) for details on the $\mathscr{H}_\infty$ control problem and related discussions.

3.1 The Choice of Filter

For the stable plant case, the low-pass filter F can have the form of equation (4). However, plants with right-half-plane poles and/or zeros impose fundamental limitations on the achievable closed-loop performance and may be difficult to handle (Freudenberg and Looze, 1985). Suppose the plant P has closed-right-half-plane poles at p_i and closed-right-half-plane zeros at z_i, in which case $T = \frac{PC}{1+PC}$ will have to satisfy the following analytic constraints

$$T(p_i) = 1 \quad \text{and} \quad T(z_i) = 0 \,. \tag{10}$$

These constraints ought to be at least roughly reflected in the choice of filter F since we wish to have T close to P_aF in an $\mathscr{H}_\infty$ sense [5].

Letting F have the form

$$F(s) = (b_{k-1}s^{k-1} + ... + b_1 s + b_0)\left(\frac{\lambda}{s+\lambda}\right)^{n+k-1} \tag{11}$$

where n is at least equal to the relative degree of P, the coefficients $b_0, b_1, ..., b_{k-1}$ can be chosen such that

$$[P_aF]_{s=p_i} = 1 \quad \text{and} \quad [P_aF]_{s=z_i} = 0 \,.$$

Notice that P_aF is automatically zero at the open right half-plane zeros of P since P_a contains the open right half-plane zeros of P. Thus the second condition reduces to simply requiring

$$[P_aF]_{s=j\omega_i} = 0$$

where $j\omega_i$ are the $j\omega$-axis zeros of P, if any.

Furthermore, in a typical design we will generally wish zero steady-state error for a unit step response. This corresponds to requiring

$$T(0) = 1 \,.$$

Hence, in summary we shall select the filter F as in equation (11) with coefficients $b_0, b_1, ..., b_{k-1}$ chosen such that

$$[P_aF]_{s=p_i,0} = 1 \quad \text{and} \quad [P_aF]_{s=j\omega_i} = 0 \,.$$

[5] In fact, we can have P_aF precisely satisfy the constraints in (10) if we wish, though $|P_aF|$ may not then be of an attractive shape.

For these sets of constraints to be solvable, one needs to introduce as many coefficients b_j as constraints. Hence we select k in equation (11) to be equal to the number of analytic constraints.

3.2 Design of weighting functions $\varepsilon_1(s)$ and $\varepsilon_2(s)$

Weighting functions $\varepsilon_1(s)$ and $\varepsilon_2(s)$ were introduced as a part of the $\mathscr{H}_\infty$ index in Equation (9) to achieve the desired effect detailed below. We will have different objectives in different frequency regions based upon the particular application specifications and also the characteristics of the plant. Let us now set out our design objectives, as specified in the index in Equation (9):

i. Let α be the desired closeness between $PC/(1+PC)$ and P_aF in an $\mathscr{H}_\infty$ sense. That is, we require $\|PC/(1+PC) - P_aF\|_\infty \le \alpha$.
ii. Let β_p^i be the maximum tolerable gain in the appropriate frequency region for the transfer function $T_{yr_2} = P/(1+PC)$. That is, we require $\bar{\sigma}\,[P/(1+PC)(j\omega)] \le \beta_p^i \quad \forall\omega \in [\omega_1^i, \omega_2^i]$.
iii. Let β_c^i be the maximum tolerable gain in the appropriate frequency region for the transfer function $T_{ur_1} = C/(1+PC)$. That is, we require $\bar{\sigma}\,[C/(1+PC)(j\omega)] \le \beta_c^i \quad \forall\omega \in [\omega_3^i, \omega_4^i]$.

Now, we have one number, α, and two sets of different numbers, namely β_p^i and β_c^i, that represent our objectives. These numbers will be used to specify $\varepsilon_1(s)$ and $\varepsilon_2(s)$ as we discuss next. Once $\varepsilon_1(s)$ and $\varepsilon_2(s)$ are specified, we just need to check the number γ to determine whether the design was successful in achieving our objectives or not. Towards this end, note that the index in (9) certainly guarantees that

$$\bar{\sigma}\left[\tfrac{PC}{1+PC} - P_aF\right] \le \gamma \qquad \forall\omega \,, \tag{12}$$

$$\bar{\sigma}\left[\tfrac{P}{1+PC}(j\omega)\right] \le \tfrac{\gamma}{|\varepsilon_2(j\omega)|} \qquad \forall\omega \,, \tag{13}$$

$$\bar{\sigma}\left[\tfrac{C}{1+PC}(j\omega)\right] \le \tfrac{\gamma}{|\varepsilon_1(j\omega)|} \qquad \forall\omega \tag{14}$$

are achieved. Consequently, choosing

$$|\varepsilon_1(j\omega)| \ge \tfrac{\alpha}{\beta_c^i} \quad \forall\omega \in [\omega_3^i, \omega_4^i]$$

and

$$|\varepsilon_2(j\omega)| \ge \tfrac{\alpha}{\beta_p^i} \quad \forall\omega \in [\omega_1^i, \omega_2^i]$$

will do the trick since $\gamma \le \alpha$ will mean that our three objectives above are satisfied.

3.3 Specifying $\varepsilon_1(j\omega)$ and $\varepsilon_2(j\omega)$

We shall outline four different scenarios that simply specify how $\varepsilon_1(j\omega)$ and $\varepsilon_2(j\omega)$ ought to be chosen.

If $\varepsilon_1(j\omega) = 0$ and $\varepsilon_2(j\omega) = 0$, then the $\mathscr{H}_\infty$ index specified in (9) reduces to

$$\gamma = \min_{C \in \underline{\mathcal{C}}} \|PC/(1+PC) - P_aF\|_\infty$$

Hence C will be exactly the IMC controller described in Section 2, provided that P is stable and has no $j\omega$-axis zeros and all other assumptions of the IMC design method outlined in Section 2 are fulfilled (i.e. $\gamma = 0$ for such a case).

If $\varepsilon_1(j\omega) = 0$ and $\varepsilon_2(j\omega) \neq 0$, then the $\mathscr{H}_\infty$ index specified in (9) reduces to

$$\gamma = \min_{C \in \underline{\mathcal{C}}} \left\| \frac{PC}{1+PC} - P_a F \quad \varepsilon_2(s)\frac{P}{1+PC} \right\|_\infty$$

It was earlier discussed in Section 2 that if P had for example lightly-damped poles in the closed-loop bandwidth, then an IMC controller would result in a transfer function, $P/(1 + PC)$, that has large gain near the frequency of the lightly-damped poles. It was also explained that this is highly undesirable in a sensible design. Consequently, we choose $|\varepsilon_2(j\omega)| \geq \alpha/\beta_p^i$ near the frequencies of the lightly-damped poles of P, as this will then limit the size of $P/(1 + PC)$ and we are free to let $|\varepsilon_2(j\omega)|$ become small at frequencies far away from the lightly-damped poles of P. Hence $\gamma \leq \alpha$ will imply $\bar{\sigma}\left[P/(1+PC)(j\omega) - P_aF(j\omega)\right] \leq \alpha \quad \forall\omega$ and $\bar{\sigma}\left[P/(1+PC)(j\omega)\right] \leq \beta_p^i \quad \forall\omega : |\varepsilon_2(j\omega)| \geq \alpha/\beta_p^i$. Therefore, the closeness of $PC/(1 + PC)$ to P_aF is traded-off with limiting the size of $P/(1 + PC)$ at the problematic frequencies.

It is now clear how $\varepsilon_1(j\omega)$and $\varepsilon_2(j\omega)$ can also be specified for the other two cases where $(\varepsilon_1(j\omega) \neq 0, \varepsilon_2(j\omega) = 0)$ and $(\varepsilon_1(j\omega) \neq 0, \varepsilon_2(j\omega) \neq 0)$.

4. THE PROPOSED $\mathscr{H}_\infty$ CONTROL DESIGN PROCEDURE

Let us now summarize the proposed $\mathscr{H}_\infty$ control design method.

- **Step 1.** Given a model of the true plant P, do the decomposition in Equation (8);
- **Step 2.** Choose an appropriate transfer function for the filter F according to the model P and the discussions in Section 3.1;
- **Step 3.** Find the critical frequency regions where P has lightly-damped poles, lightly-damped zeros or frequencies above the bandwidth of P but below the bandwidth of F. Based on the desired closed-loop objectives and specifications, set the positive numbers α, β_p^i and β_c^i according to the discussions in Section 3.2;
- **Step 4.** Design the frequency weights, $\varepsilon_1(s)$ and $\varepsilon_2(s)$, according to the rules given in Sections 3.2 and 3.3, using the specified values α, β_p^i and β_c^i in Step 3, for the appropriate frequency regions;
- **Step 5.** Solve the $\mathscr{H}_\infty$ controller design problem given in Equation (9) and obtain γ and the admissible controller C;
- **Step 6.** If $\gamma \leq \alpha$, the obtained controller C achieves the desired performance objectives specified in Step 3.

In the following section we shall use the above-stated procedure to show the effectiveness and easy-to-use features of our proposed $\mathscr{H}_\infty$ design method.

5. NUMERICAL EXAMPLE

We shall consider an example in order to illustrate the advantages and effectiveness of the proposed design method. This example illustrates the difficulty with the IMC approach discussed in Section 2 where we argued that if the plant model P has lightly-damped stable/unstable zeros within the closed-loop passband, then the (2,1) entry of $H(P,C)$ in (1) will grow large near the frequencies of those zeros and hence the control signal in Fig. 1 will be large.

Consider a plant model which has the form

$$P_1 = \frac{1/9(s^2 + 0.01s + 9)}{(s+1)^3}$$

with lightly-damped zeros at $s = -0.005 \pm j3$.

Following the standard IMC design procedure for stable plants discussed in Section 2, the controller that achieves a closed-loop transfer function with bandwidth of $0.3\,rad/s$ is $C_1 = 2.7(s+1)^3/s(s^2 + 0.01s+9)$. Notice that the (2,1) entry of $H(P_1, C_1)$ in (1), $T_{yr_1} = C_1/(1+P_1C_1)$, has a maximum gain of 712 ($\simeq 57dB$) near the frequency of the lightly-damped zeros ($3\,rad/s$) as shown in Fig. 2. This is undesirable for a sensible design.

Given that the lightly-damped zero is at $10\times$ the closed-loop bandwidth, it is unfortunately clear that sensible design should be readily achievable. All the damage is being caused by the IMC method, as oppose to a plant that is intrinsically difficult to control.

With this difficulty in mind, let us employ our new $\mathscr{H}_\infty$ algorithm and show how we fix this drawback of the IMC design method. Following the algorithm outlined in Section 3, P_1 is decomposed into $[P_1]_a = 1$ and $[P_1]_m = (s^2 + 0.01s + 9)/9(s + 1)^3$ and we set our closed-loop performance objectives as in Step 3 of our proposed controller design procedure. We seek to: (a) have the closeness between $P_1C_1/(1 + P_1C_1)$ and $[P_1]_aF_1$ in an $\mathscr{H}_\infty$ sense below 0.1 ($\alpha = 0.1$); (b) keep T_{yr_1} below $\beta_c = 18$ ($\approx 25\,dB$). Here we assume that the actuators can pump up a maximum gain of 18, which is by a factor of 40 less than the gain (712) the actuators have to pump up if C_1 is used.

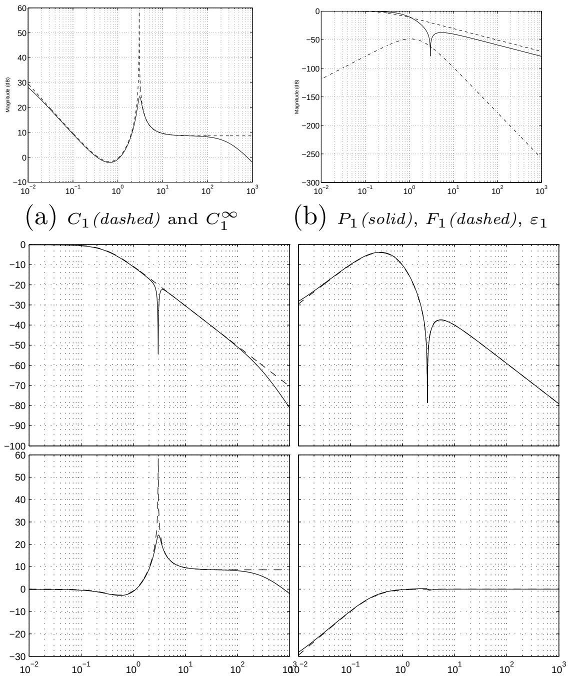
Based on the rules stated in Section 3.2 and the aforementioned objectives, the frequency cost $\varepsilon_1(j\omega)$ is designed (Fig. 2) and $\varepsilon_2(j\omega)$ is set to zero. Solving the $\mathscr{H}_\infty$ index in (9) with the same $F_1 = 0.3/(s + 0.3)$ as the desired filter transfer function we find

$$\tilde{C}_1^\infty(s) = 827.075 \quad \frac{\prod_{i=1}^{6}(s - z_i)}{\prod_{i=1}^{7}(s - p_i)}$$

with its zeros and poles given in Table 1.

Table 1. Poles and Zeros of $\tilde{C}_1^\infty$

z_i	p_i
-1.796	-306.12
-1.056	-0.005
$-1.5860 \pm j0.257$	-1.825
$-0.9719 \pm j0.0473$	$-0.29 \pm j3.045$
	$-1.57 \pm j0.266$



(a) C_1 *(dashed)* and C_1^∞ (b) P_1 *(solid)*, F_1 *(dashed)*, ε_1

(c) Four entries of $H(P_1, C_1)$ *(dashed)* and $H(P_1, C_1^\infty)$

Fig. 2. Magnitude Responses

The norm $\gamma_1 = 0.098 < \alpha$ which means that our desired closed-loop objectives ($\alpha = 0.1$, $\beta_c = 18$) have been achieved (Section 3.2). We employ the closed-loop controller reduction method detailed in Section 4.3 (pp 137–140) of reference (Obinata and Anderson, 2001) and assume that the Hankel singular values of the graph symbol of the controller are decreasingly ordered ($\sigma_1 > \sigma_2 > \sigma_3 > .... > \sigma_7$). We perform balanced realization and the result is truncated to retain all Hankel singular values greater than $0.01\,\sigma_1$. The resulting controller after truncation is

$$C_1^\infty = \frac{827.07(s+0.935)(s^2+2.066s+1.07)}{(s+306.1)(s+0.0055)(s^2+0.581s+9.361)}$$

The frequency responses of C_1 and C_1^∞, P_1, $\varepsilon_1(j\omega)$ and F_1, and $H(P_1, C_1)$ and $H(P_1, C_1^\infty)$ are plotted in Fig. 2.

The above example showed one of many attributes of our proposed design procedure. Furthermore, there are situations (see Section 2) for which it is possible to design an IMC controller, but the designed controller leads to poor performance, unlike in our proposed $\mathscr{H}_\infty$ criterion.

Our proposed $\mathscr{H}_\infty$ design method provides assurance and reliability that the controller obtained meets the pre-specified performance specifications and objectives through checking our simple flag γ.

6. CONCLUSIONS

In this paper, we have introduced a new controller design method which inherits the desirable features of the IMC but extends its applicability. Section 2 reaffirms that there exist distinct procedures for stable and unstable plants; each with its own limitations and restrictive assumptions. The proposed method uses an $\mathscr{H}_\infty$ control design method, in which the shortcomings and deficiencies of the previously used IMC design methods (Morari and Zafiriou, 1989), are addressed. Moreover, this $\mathscr{H}_\infty$ design method can be used for stable or unstable plants or plants with $j\omega$-axis zeros. In addition, it handles well plants with lightly-damped poles and zeros and situations where the bandwidth of F is orders of magnitude greater than that of P. This algorithm also gives us one number, γ, that easily flags whether the desired performance specifications have been achieved. An extension of this research is underway to deal with MIMO systems.

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