FUZZY AND NEURAL CONTROL
DISC Course Lecture Notes (October 2001)

ROBERT BABUŠKA
Control Engineering Laboratory
Faculty of Information Technology and Systems
Delft University of Technology, Delft, the Netherlands

Delft University of Technology
Delft, the Netherlands

Contents

1 INTRODUCTION

This chapter gives a brief introduction to the subject of the book and presents an outline of the different chapters. Included is also information about the expected background of the reader. Finally, the WWW and MATLAB support of the present material is described.

1.1 Intelligent Control

Conventional control engineering approaches are based on mathematical models, typically using differential and difference equations. For such models, mathematical methods and procedures for the design, formal analysis and verification of control systems have been developed. These methods, however, can only be applied to a relatively narrow class of model structures, including linear models and some specific types of nonlinear models.

Practical application of classical control design typically falls short in the situation when no mathematical model of the process to be controlled is available, or when it is nonlinear to such a degree that the available techniques cannot be applied.

This led scientists to the search for alternative modeling and control paradigms and to the introduction of "intelligent" control. Intelligent methodologies employ biologically motivated techniques and procedures to develop models of reality and to design controllers for dynamic systems. They use alternative representation schemes, such as natural language, rules, semantic networks or qualitative models, and possess formal methods to incorporate extra relevant information that conventional control cannot handle (such as heuristic knowledge of process operators). Fuzzy control is an example of a rule-based representation of human knowledge and deductive processes. Artificial neural networks, on the other hand, realize learning and adaptation capabilities by imitating the functioning of biological neural systems. With the advances in the data processing and computer technology, large amounts of process data are becoming available. This makes it possible to combine knowledge-based control with effective data driven techniques for the acquisition of models and tuning of controllers.

1.2 Organization of the Book

The material is organized in eight chapters. In Chapter 2, the basics of fuzzy set theory are explained. Chapter 3 then presents various types of fuzzy systems and their application in dynamic modeling. Fuzzy set techniques can be useful in data analysis and pattern recognition. To this end, Chapter 4 presents the basic concepts of fuzzy clustering, which can be used as one of data-driven techniques for the construction of fuzzy models from data. These data-driven construction techniques are addressed in Chapter 5. Controllers can also be design without using a process model. Chapter 6 is devoted to model-free knowledge-based design of fuzzy controllers. In Chapter 7, artificial neural networks are explained in terms of their architectures and training methods. Neural and fuzzy models can be used to design controller or can become a part of a model-based control scheme, as explained in Chapter 8.

Three appendices have been included that provide background on ordinary set theory (Appendix A), MATLAB code of some of the presented methods and algorithms (Appendix B) and a list of symbols used throughout the text (Appendix C).

It has been one of the author's aims to present the new material (fuzzy end neural techniques) in such a way that no prior knowledge about these subjects is necessary for understanding the text. It is assumed, however, that the reader has some basic knowledge of mathematical analysis (univariate and multivariate functions), linear algebra (system of linear equations, least-square solution) and systems and control theory (dynamic systems, state-feedback, PID control, linearization).

1.3 WEB and Matlab Support

The material presented in the book is supported by a WEB page containing the basic information about the course: `http://lcewww.et.tudelft.nl/~discfuzz`. MATLAB tools, demos and the overhead sheets used in the lectures can be downloaded from the page.

1.4 Acknowledgement

I wish to express my sincere thanks to my colleagues Janos Abonyi and Stanimir Mollov who read parts of the manuscript and contributed by their comments and suggestions.

2 FUZZY SETS AND RELATIONS

This chapter provides a basic introduction to fuzzy sets, fuzzy relations and operations with fuzzy sets. For a more comprehensive treatment see, for instance, (Klir and Folger, 1988; Zimmermann, 1996; Klir and Yuan, 1995).

Zadeh (1965) introduced fuzzy set theory as a mathematical discipline, although the underlying ideas had already been recognized earlier by philosophers and logicians (Pierce, Russel, Łukasiewicz, among others). A comprehensive overview is given in the introduction of the "Readings in Fuzzy Sets for Intelligent Systems", edited by Dubois, Prade and Yager (1993). A broader interest in fuzzy sets started in the seventies with their application to control and other technical disciplines.

2.1 Fuzzy Sets

In ordinary (non fuzzy) set theory, elements either fully belong to a set or are fully excluded from it. Recall, that the membership $\mu_A(x)$ of x of a classical set A, as a subset of the universe X, is defined by:[1]

$$\mu_A(x) = \begin{cases} 1, & \text{iff} \quad x \in A, \\ 0, & \text{iff} \quad x \notin A\,. \end{cases} \tag{2.1}$$

This means that an element x is either a member of set A ($\mu_A(x) = 1$) or not ($\mu_A(x) = 0$). This strict classification is useful in the mathematics and other sciences

[1] A brief summary of basic concepts related to ordinary sets is given in Appendix A.

that rely on precise definitions. Ordinary set theory complements bi-valent logic in which a statement is either true or false. While in mathematical logic the emphasis is on preserving formal validity and truth under any and every interpretation, in many real-life situations and engineering problems, the aim is to preserve information in the given context. In these situations, it may not be quite clear whether an element belongs to a set or not.

For example, if set A represents PCs which are too expensive for a student's budget, then it is obvious that this set has no clear boundaries. Of course, it could be said that a PC priced at $2500 is too expensive, but what about PCs priced at $2495 or $2502? Are those PCs too expensive or not? Clearly, a boundary could be determined above which a PC is too expensive for the average student, say $2500, and a boundary below which a PC is certainly not too expensive, say $1000. Between those boundaries, however, there remains a vague interval in which it is not quite clear whether a PC is too expensive or not. In this interval, a grade could be used to classify the price as partly too expensive. This is where fuzzy sets come in: sets of which the membership has grades in the unit interval [0,1].

A fuzzy set is a set with graded membership in the real interval: $\mu_A(x) \in [0, 1]$. That is, elements can belong to a fuzzy set to a certain degree. As such, fuzzy sets can be used for mathematical representations of vague concepts, such as *low temperature*, *fairly tall person*, *expensive car*, etc.

Definition 2.1 (Fuzzy Set) *A fuzzy set A on universe (domain) X is a set defined by the membership function $\mu_A(x)$ which is a mapping from the universe X into the unit interval:*

$$\mu_A(x)\colon X \to [0, 1] \,. \tag{2.2}$$

$\mathcal{F}(X)$ denotes the set of all fuzzy sets on X.

If the value of the membership function, called the membership degree (or grade), equals one, x belongs completely to the fuzzy set. If it equals zero, x does not belong to the set. If the membership degree is between 0 and 1, x is a partial member of the fuzzy set:

$$\mu_A(x) \begin{cases} = 1 & x \text{ is a full member of } A \\ \in (0, 1) & x \text{ is a partial member of } A \\ = 0 & x \text{ is not member of } A \end{cases} \tag{2.3}$$

In the literature on fuzzy set theory, ordinary (nonfuzzy) sets are usually referred to as *crisp (or hard) sets*. Various symbols are used to denote membership functions and degrees, such as $\mu_A(x)$, $A(x)$ or just a.

Example 2.1 (Fuzzy Set) Figure 2.1 depicts a possible membership function of a fuzzy set representing PCs *too expensive* for a student's budget.

According to this membership function, if the price is below $1000 the PC is certainly not too expensive, and if the price is above $2500 the PC is fully classified as too expensive. In between, an increasing membership of the fuzzy set *too expensive* can be seen. It is not necessary that the membership linearly increases with the price, nor that there is a non-smooth transition from $1000 to $2500. Note that in engineering

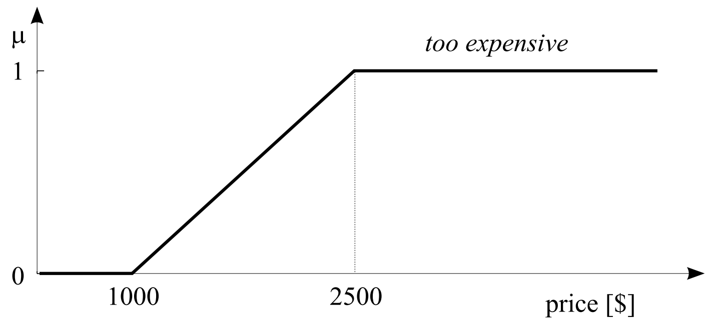

Figure 2.1. Fuzzy set A representing PCs too expensive for a student's budget.

applications the choice of the membership function for a fuzzy set is rather arbitrary. □

2.2 Properties of Fuzzy Sets

To establish the mathematical framework for computing with fuzzy sets, a number of properties of fuzzy sets need to be defined. This section gives an overview of only the ones that are strictly needed for the rest of the book. They include the definitions of the height, support, core, α-cut and cardinality of a fuzzy set. In addition, the properties of normality and convexity are introduced. For a more complete treatment see (Klir and Yuan, 1995).

2.2.1 Normal and Subnormal Fuzzy Sets

We learned that the membership of elements in fuzzy sets is a matter of degree. The *height* of a fuzzy set is the largest membership degree among all elements of the universe. Fuzzy sets whose height equals one for at least one element x in the domain X are called *normal* fuzzy sets. The height of *subnormal* fuzzy sets is thus smaller than one for all elements in the domain. Formally we state this by the following definitions.

Definition 2.2 (Height) *The height of a fuzzy set A is the supremum of the membership grades of elements in A:*

$$\operatorname{hgt}(A) = \sup_{x \in X} \mu_A(x)\,. \tag{2.4}$$

For a discrete domain X, the supremum (the least upper bound) becomes the maximum and hence the height is the largest degree of membership for all $x \in X$.

Definition 2.3 (Normal Fuzzy Set) *A fuzzy set A is* normal *if $\exists x \in X$ such that $\mu_A(x) = 1$. Fuzzy sets that are not normal are called* subnormal. *The operator $\operatorname{norm}(A)$ denotes normalization of a fuzzy set, i.e., $A' = \operatorname{norm}(A) \Leftrightarrow \mu_{A'}(x) = \mu_A(x)/\operatorname{hgt}(A),\ \forall x$.*

2.2.2 Support, Core and α-cut

Support, core and α-cut are *crisp* sets obtained from a fuzzy set by selecting its elements whose membership degrees satisfy certain conditions.

Definition 2.4 (Support) *The* support *of a fuzzy set A is the crisp subset of X whose elements all have nonzero membership grades:*

$$\text{supp}(A) = \{x \mid \mu_A(x) > 0\}\,. \tag{2.5}$$

Definition 2.5 (Core) *The core of a fuzzy set A is a crisp subset of X consisting of all elements with membership grades equal to one:*

$$\text{core}(A) = \{x \mid \mu_A(x) = 1\}\,. \tag{2.6}$$

In the literature, the core is sometimes also denoted as the kernel, ker(A). The core of a subnormal fuzzy set is empty.

Definition 2.6 (α-Cut) *The α-cut A_α of a fuzzy set A is the crisp subset of the universe of discourse X whose elements all have membership grades greater than or equal to α:*

$$A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}, \quad \alpha \in [0, 1]\,. \tag{2.7}$$

The α-cut operator is also denoted by α-cut(A) or α-cut(A, α). An α-cut A_α is strict if $\mu_A(x) \neq \alpha$ for each $x \in A_\alpha$. The value α is called the α-level.

Figure 2.2 depicts the core, support and α-cut of a fuzzy set.

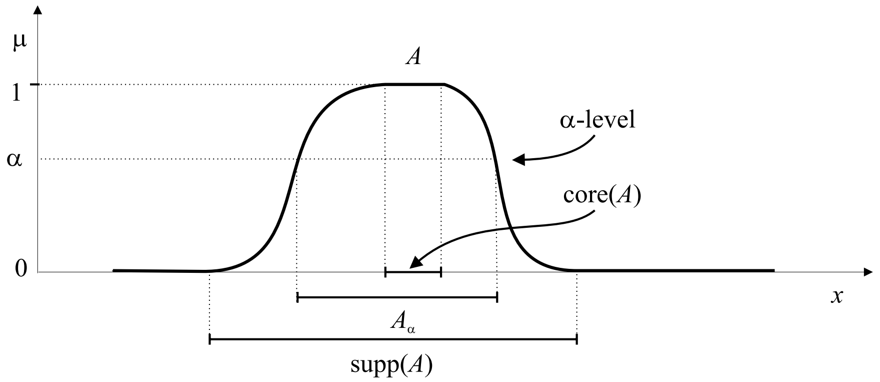

Figure 2.2. Core, support and α-cut of a fuzzy set.

The core and support of a fuzzy set can also be defined by means of α-cuts:

$$\text{core}(A) = 1\text{-cut}(A) \tag{2.8}$$
$$\text{supp}(A) = 0\text{-cut}(A) \tag{2.9}$$

2.2.3 Convexity and Cardinality

Membership function may be unimodal (with one global maximum) or multimodal (with several maxima). Unimodal fuzzy sets are called convex fuzzy sets. Convexity can also be defined in terms of α-cuts:

Definition 2.7 (Convex Fuzzy Set) *A fuzzy set defined in $\mathbb{R}^n$ is convex if each of its α-cuts is a convex set.*

Figure 2.3 gives an example of a convex and non-convex fuzzy set.

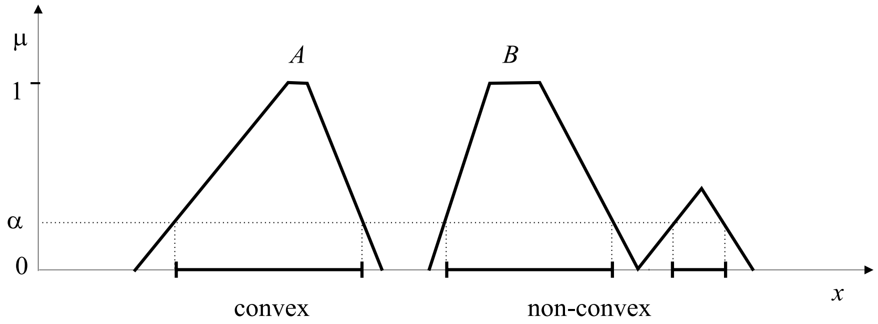

Figure 2.3. The core of a non-convex fuzzy set is a non-convex (crisp) set.

Example 2.2 (Non-convex Fuzzy Set) Figure 2.4 gives an example of a non-convex fuzzy set representing “high-risk age” for a car insurance policy. Drivers who are too young or too old present higher risk than middle-aged drivers.

□

Definition 2.8 (Cardinality) *Let $A = \{\mu_A(x_i) \mid i = 1, 2, \ldots, n\}$ be a finite discrete fuzzy set. The cardinality of this fuzzy set is defined as the sum of the membership*

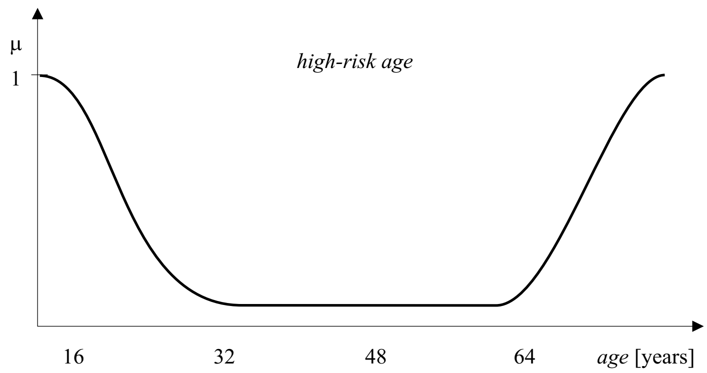

Figure 2.4. A fuzzy set defining “high-risk age” for a car insurance policy is an example of a non-convex fuzzy set.

degrees:

$$|A| = \sum_{i=1}^{n} \mu_A(x_i)\,. \tag{2.11}$$

2.3 Representations of Fuzzy Sets

There are several ways to define (or represent in a computer) a fuzzy set: through an analytic description of its membership function $\mu_A(x) = f(x)$, as a list of the domain elements end their membership degrees or by means of α-cuts. These possibilities are discussed below.

2.3.1 Similarity-based Representation

Fuzzy sets are often defined by means of the (dis)similarity of the considered object x to a given prototype v of the fuzzy set

$$\mu(x) = \frac{1}{1 + d(x, v)}\,. \tag{2.12}$$

Here, $d(x, v)$ denotes a dissimilarity measure which in metric spaces is typically a distance measure (such as the Euclidean distance). The prototype is a full member (typical element) of the set. Elements whose distance from the prototype goes to zero have membership grades close to one. As the distance grows, the membership decreases. As an example, consider the membership function:

$$\mu_A(x) = \frac{1}{1 + x^2}, \quad x \in \mathbb{R},$$

representing "approximately zero" real numbers.

2.3.2 Parametric Functional Representation

Various forms of parametric membership functions are often used:

- *Trapezoidal* membership function:

  $$\mu(x; a, b, c, d) = \max\left(0, \min\left(\frac{x - a}{b - a}, 1, \frac{d - x}{d - c}\right)\right), \tag{2.13}$$

  where a, b, c and d are the coordinates of the trapezoid's apexes. When $b = c$, a *triangular* membership function is obtained.

- *Piece-wise exponential* membership function:

  $$\mu(x; c_l, c_r, w_l, w_r) = \begin{cases} \exp(-(\frac{x - c_l}{2w_l})^2), & \text{if } x < c_l, \\ \exp(-(\frac{x - c_r}{2w_r})^2), & \text{if } x > c_r, \\ 1, & \text{otherwise}, \end{cases} \tag{2.14}$$

  where c_l and c_r are the left and right shoulder, respectively, and w_l, w_r are the left and right width, respectively. For $c_l = c_r$ and $w_l = w_r$ the Gaussian membership function is obtained.

Figure 2.5 shows examples of triangular, trapezoidal and bell-shaped (exponential) membership functions. A special fuzzy set is the *singleton set* (fuzzy set representation of a number) defined by:

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = x_0, \\ 0, & \text{otherwise}. \end{cases} \tag{2.15}$$

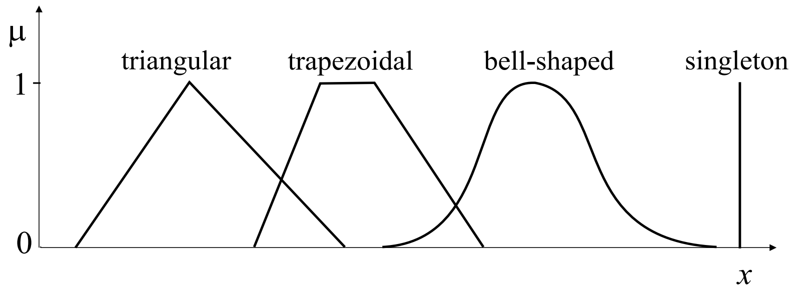

Figure 2.5. Different shapes of membership functions.

Another special set is the *universal set*, whose membership function equals one for all domain elements:

$$\mu_A(x) = 1, \quad \forall x. \tag{2.16}$$

Finally, the term *fuzzy number* is sometimes used to denote a normal, convex fuzzy set which is defined on the real line.

2.3.3 Point-wise Representation

In a discrete set $X = \{x_i \mid i = 1, 2, \ldots, n\}$, a fuzzy set A may be defined by a list of ordered pairs: membership degree/set element:

$$A = \{\mu_A(x_1)/x_1,\ \mu_A(x_2)/x_2,\ \ldots, \mu_A(x_n)/x_n\} = \{\mu_A(x)/x \mid x \in X\}, \tag{2.17}$$

Normally, only elements $x \in X$ with non-zero membership degrees are listed. The following alternatives to the above notation can be encountered:

$$A = \mu_A(x_1)/x_1 + \cdots + \mu_A(x_n)/x_n = \sum_{i=1}^{n} \mu_A(x_i)/x_i \tag{2.18}$$

for finite domains, and

$$A = \int_X \mu_A(x)/x \tag{2.19}$$

for continuous domains. Note that rather than summation and integration, in this context, the $\sum$, $+$ and $\int$ symbols represent a collection (union) of elements.

A pair of vectors (arrays in computer programs) can be used to store discrete membership functions:

$$\mathbf{x} = [x_1, x_2, \ldots, x_n], \quad \boldsymbol{\mu} = [\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)]. \tag{2.20}$$

Intermediate points can be obtained by interpolation. This representation is often used in commercial software packages. For an equidistant discretization of the domain it is sufficient to store only the membership degrees μ.

2.3.4 Level Set Representation

A fuzzy set can be represented as a list of α levels ($\alpha \in [0, 1]$) and their corresponding α-cuts:

$$A = \{\alpha_1/A_{\alpha_1},\ \alpha_2/A_{\alpha_2},\ \ldots, \alpha_n/A_{\alpha_n}\} = \{\alpha/A_{\alpha_n} \mid \alpha \in (0, 1)\}, \qquad (2.21)$$

The range of α must obviously be discretized. This representation can be advantageous as operations on fuzzy subsets of the same universe can be defined as classical set operations on their level sets. Fuzzy arithmetic can thus be implemented by means of interval arithmetic, etc. In multidimensional domains, however, the use of the level-set representation can be computationally involved.

Example 2.3 (Fuzzy Arithmetic) Using the level-set representation, results of arithmetic operations with fuzzy numbers can be obtained as a collection standard arithmetic operations on their α-cuts. As an example consider addition of two fuzzy numbers A and B defined on the real line:

$$A + B = \{\alpha/(A_{\alpha_n} + B_{\alpha_n}) \mid \alpha \in (0, 1)\}, \qquad (2.22)$$

where $A_{\alpha_n} + B_{\alpha_n}$ is the addition of two intervals.

□

2.4 Operations on Fuzzy Sets

Definitions of set-theoretic operations such as the complement, union and intersection can be extended from ordinary set theory to fuzzy sets. As membership degrees are no longer restricted to $\{0, 1\}$ but can have any value in the interval $[0, 1]$, these operations cannot be uniquely defined. It is clear, however, that the operations for fuzzy sets must give correct results when applied to ordinary sets (an ordinary set can be seen as a special case of a fuzzy set).

This section presents the basic definitions of fuzzy intersection, union and complement, as introduced by Zadeh. General intersection and union operators, called triangular norms (t-norms) and triangular conorms (t-conorms), respectively, are given as well. In addition, operations of projection and cylindrical extension, related to multidimensional fuzzy sets, are given.

2.4.1 Complement, Union and Intersection

Definition 2.9 (Complement of a Fuzzy Set) *Let A be a fuzzy set in X. The complement of A is a fuzzy set, denoted $\bar{A}$, such that for each $x \in X$:*

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)\,. \qquad (2.23)$$

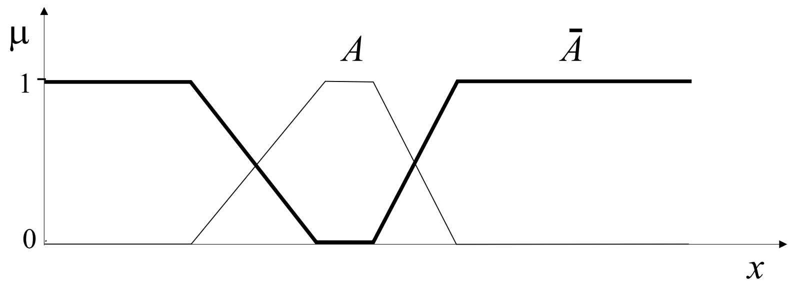

Figure 2.6. Fuzzy set and its complement $\bar{A}$ in terms of their membership functions.

Figure 2.6 shows an example of a fuzzy complement in terms of membership functions. Besides this operator according to Zadeh, other complements can be used. An example is the λ-complement according to Sugeno (1977):

$$\mu_{\bar{A}}(x) = \frac{1 - \mu_A(x)}{1 + \lambda \mu_A(x)} \tag{2.24}$$

where $\lambda > 0$ is a parameter.

Definition 2.10 (Intersection of Fuzzy Sets) *Let A and B be two fuzzy sets in X. The intersection of A and B is a fuzzy set C, denoted $C = A \cap B$, such that for each $x \in X$:*

$$\mu_C(x) = \min[\mu_A(x), \mu_B(x)] \,. \tag{2.25}$$

The minimum operator is also denoted by '$\wedge$', i.e., $\mu_C(x) = \mu_A(x) \wedge \mu_B(x)$. Figure 2.7 shows an example of a fuzzy intersection in terms of membership functions.

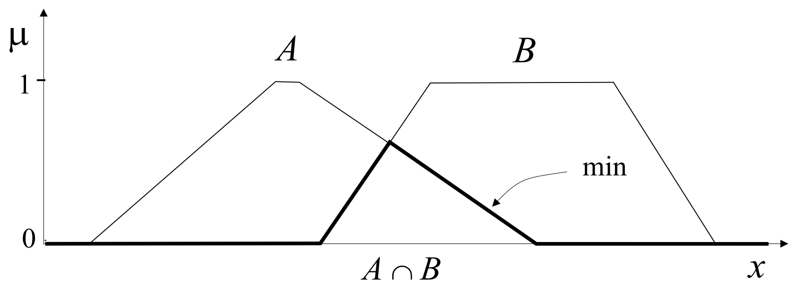

Figure 2.7. Fuzzy intersection $A \cap B$ in terms of membership functions.

Definition 2.11 (Union of Fuzzy Sets) *Let A and B be two fuzzy sets in X. The union of A and B is a fuzzy set C, denoted $C = A \cup B$, such that for each $x \in X$:*

$$\mu_C(x) = \max[\mu_A(x), \mu_B(x)] \,. \tag{2.26}$$

The maximum operator is also denoted by '$\vee$', i.e., $\mu_C(x) = \mu_A(x) \vee \mu_B(x)$. Figure 2.8 shows an example of a fuzzy union in terms of membership functions.

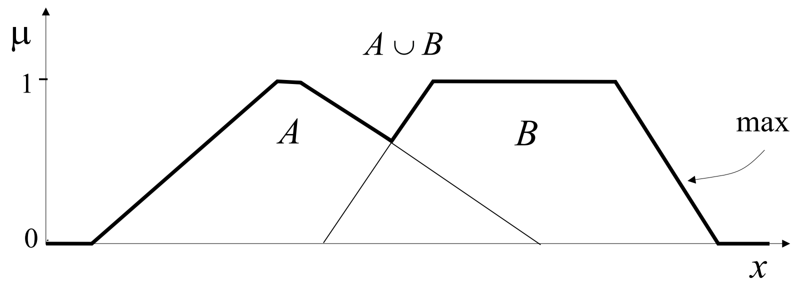

Figure 2.8. Fuzzy union $A \cup B$ in terms of membership functions.

2.4.2 T-norms and T-conorms

Fuzzy intersection of two fuzzy sets can be specified in a more general way by a binary operation on the unit interval, i.e., a function of the form:

$$T \colon [0,1] \times [0,1] \to [0,1] \tag{2.27}$$

In order for a function T to qualify as a fuzzy intersection, it must have appropriate properties. Functions known as t-norms (triangular norms) posses the properties required for the intersection. Similarly, functions called t-conorms can be used for the fuzzy union.

Definition 2.12 (t-Norm/Fuzzy Intersection) *A t-norm T is a binary operation on the unit interval that satisfies at least the following axioms for all $a, b, c \in [0,1]$ (Klir and Yuan, 1995):*

$$\begin{array}{ll} T(a,1) = a & \textit{(boundary condition)}, \\ b \le c \ \textit{implies} \ \ T(a,b) \le T(a,c) & \textit{(monotonicity)}, \\ T(a,b) = T(b,a) & \textit{(commutativity)}, \\ T(a,T(b,c)) = T(T(a,b),c) & \textit{(associativity)}\,. \end{array} \tag{2.28}$$

Some frequently used t-norms are:

standard (Zadeh) intersection: $T(a,b) = \min(a,b)$
algebraic product (probabilistic intersection): $T(a,b) = ab$
Łukasiewicz (bold) intersection: $T(a,b) = \max(0, a+b-1)$

The minimum is the largest t-norm (intersection operator). For our example shown in Figure 2.7 this means that the membership functions of fuzzy intersections $A \cap B$ obtained with other t-norms are all below the bold membership function (or partly coincide with it).

Definition 2.13 (t-Conorm/Fuzzy Union) *A t-conorm S is a binary operation on the unit interval that satisfies at least the following axioms for all $a, b, c \in [0,1]$ (Klir and Yuan, 1995):*

$$\begin{array}{ll} S(a,0) = a & \textit{(boundary condition)}, \\ b \le c \ \textit{implies} \ \ S(a,b) \le S(a,c) & \textit{(monotonicity)}, \\ S(a,b) = S(b,a) & \textit{(commutativity)}, \\ S(a,S(b,c)) = S(S(a,b),c) & \textit{(associativity)}\,. \end{array} \tag{2.29}$$

Some frequently used t-conorms are:

standard (Zadeh) union:	$S(a, b) = \max(a, b)$,
algebraic sum (probabilistic union):	$S(a, b) = a + b - ab$,
Łukasiewicz (bold) union:	$S(a, b) = \min(1, a + b)$.

The maximum is the smallest t-conorm (union operator). For our example shown in Figure 2.8 this means that the membership functions of fuzzy unions $A \cup B$ obtained with other t-conorms are all above the bold membership function (or partly coincide with it).

2.4.3 Projection and Cylindrical Extension

Projection reduces a fuzzy set defined in a multi-dimensional domain (such as $\mathbb{R}^2$ to a fuzzy set defined in a lower-dimensional domain (such as $\mathbb{R}$). *Cylindrical extension* is the opposite operation, i.e., the extension of a fuzzy set defined in low-dimensional domain into a higher-dimensional domain. Formally, these operations are defined as follows:

Definition 2.14 (Projection of a Fuzzy Set) *Let $U \subseteq U_1 \times U_2$ be a subset of a Cartesian product space, where U_1 and U_2 can themselves be Cartesian products of lower-dimensional domains. The projection of fuzzy set A defined in U onto U_1 is the mapping $\mathrm{proj}_{U_1}: \mathcal{F}(U) \to \mathcal{F}(U_1)$ defined by*

$$\mathrm{proj}_{U_1}(A) = \left\{ \sup_{U_2} \mu_A(u)/u_1 \middle| u_1 \in U_1 \right\} . \tag{2.30}$$

The projection mechanism eliminates the dimensions of the product space by taking the supremum of the membership function for the dimension(s) to be eliminated.

Example 2.4 (Projection) Assume a fuzzy set A defined in $U \subset X \times Y \times Z$ with $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2\}$, as follows:

$$\begin{aligned} A = \{&\mu_1/(x_1, y_1, z_1),\ \mu_2/(x_1, y_2, z_1),\ \mu_3/(x_2, y_1, z_1), \\ &\mu_4/(x_2, y_2, z_1),\ \mu_5/(x_2, y_2, z_2)\} \end{aligned} \tag{2.31}$$

Let us compute the projections of A onto X, Y and $X \times Y$:

$$\mathrm{proj}_X(A) = \{\max(\mu_1, \mu_2)/x_1,\ \max(\mu_3, \mu_4, \mu_5)/x_2\} , \tag{2.33}$$

$$\mathrm{proj}_Y(A) = \{\max(\mu_1, \mu_3)/y_1,\ \max(\mu_2, \mu_4, \mu_5)/y_2\} , \tag{2.34}$$

$$\begin{aligned} \mathrm{proj}_{X \times Y}(A) = \{&\mu_1/(x_1, y_1),\ \mu_2/(x_1, y_2), \\ &\mu_3/(x_2, y_1),\ \max(\mu_4, \mu_5)/(x_2, y_2)\} . \end{aligned} \tag{2.35}$$

□

Projections from $\mathbb{R}^2$ to $\mathbb{R}$ can easily be visualized, see Figure 2.9.

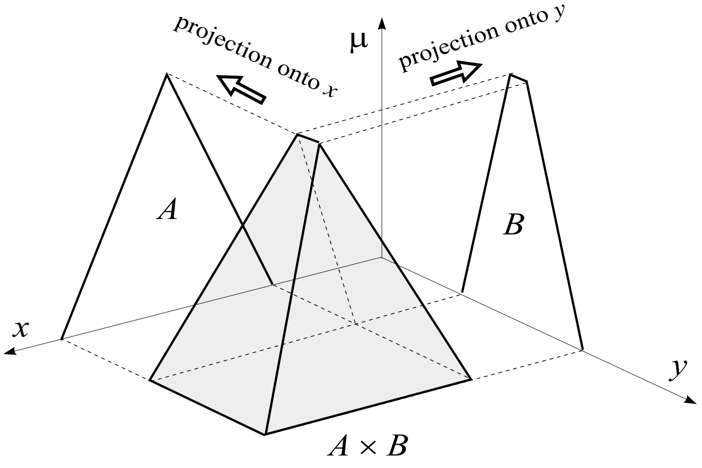

Figure 2.9. Example of projection from $\mathbb{R}^2$ to $\mathbb{R}$.

Definition 2.15 (Cylindrical Extension) *Let $U \subseteq U_1 \times U_2$ be a subset of a Cartesian product space, where U_1 and U_2 can themselves be Cartesian products of lower-dimensional domains. The cylindrical extension of fuzzy set A defined in U_1 onto U is the mapping $\text{ext}_U\colon \mathcal{F}(U_1) \to \mathcal{F}(U)$ defined by*

$$\text{ext}_U(A) = \left\{ \mu_A(u_1)/u \middle| u \in U \right\} . \tag{2.37}$$

Cylindrical extension thus simply replicates the membership degrees from the existing dimensions into the new dimensions. Figure 2.10 depicts the cylindrical extension from $\mathbb{R}$ to $\mathbb{R}^2$.

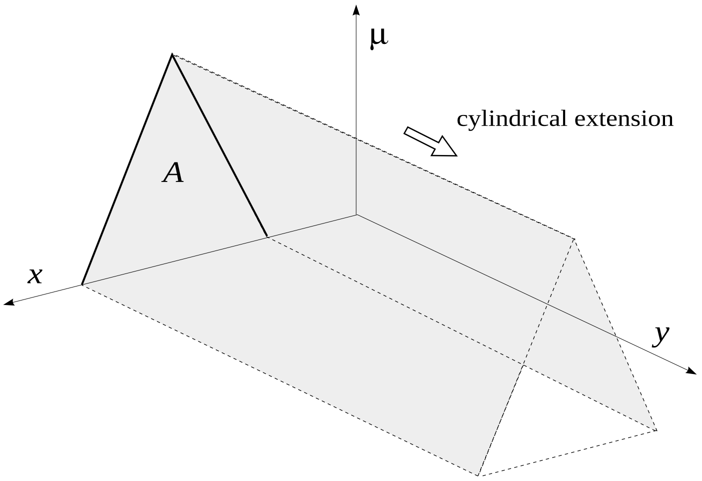

Figure 2.10. Example of cylindrical extension from $\mathbb{R}$ to $\mathbb{R}^2$.

It is easy to see that projection leads to a loss of information, thus for A defined in $X^n \subset X^m$ ($n < m$) it holds that:

$$A = \text{proj}_{X^n}(\text{ext}_{X^m}(A)), \tag{2.38}$$

but

$$A \neq \text{ext}_{X^m}(\text{proj}_{X^n}(A))\,. \tag{2.39}$$

Verify this for the fuzzy sets given in Example 2.4 as an exercise.

2.4.4 Operations on Cartesian Product Domains

Set-theoretic operations such as the union or intersection applied to fuzzy sets defined in different domains result in a multi-dimensional fuzzy set in the Cartesian product of those domains. The operation is in fact performed by first extending the original fuzzy sets into the Cartesian product domain and then computing the operation on those multi-dimensional sets.

Example 2.5 (Cartesian-Product Intersection) Consider two fuzzy sets A_1 and A_2 defined in domains X_1 and X_2, respectively. The intersection $A_1 \cap A_2$, also denoted by $A_1 \times A_2$ is given by:

$$A_1 \times A_2 = \text{ext}_{X_2}(A_1) \cap \text{ext}_{X_1}(A_2)\,. \tag{2.40}$$

This cylindrical extension is usually considered implicitly and it is not stated in the notation:

$$\mu_{A_1 \times A_2}(x_1, x_2) = \mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2)\,. \tag{2.41}$$

Figure 2.11 gives a graphical illustration of this operation.

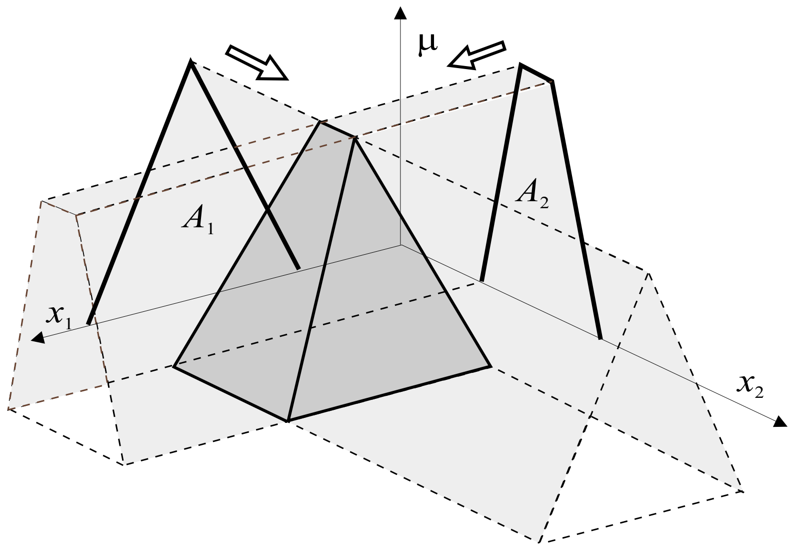

Figure 2.11. Cartesian-product intersection.

□

2.4.5 Linguistic Hedges

Fuzzy sets can be used to represent qualitative linguistic terms (notions) like “short”, “long”, “expensive”, etc. in terms of membership functions define in numerical domains (distance, price, etc.).

By means of *linguistic hedges* (linguistic modifiers) the meaning of these terms can be modified without redefining the membership functions. Examples of hedges are: *very*, *slightly*, *more or less*, *rather*, etc. Hedge “very”, for instance, can be used to change “expensive” to “very expensive”.

Two basic approaches to the implementation of linguistic hedges can be distinguished: *powered* hedges and *shifted* hedges. Powered hedges are implemented by functions operating on the membership degrees of the linguistic terms (Zimmermann, 1996). For instance, the hedge *very* squares the membership degrees of the term which meaning it modifies, i.e., $\mu_{\text{very A}}(x) = \mu_A^2(x)$. Shifted hedges (Lakoff, 1973), on the other hand, shift the membership functions along their domains. Combinations of the two approaches have been proposed as well (Novák, 1989; Novák, 1996).

Example 2.6 Consider three fuzzy sets *Small*, *Medium* and *Big* defined by triangular membership functions. Figure 2.12 shows these membership functions (solid line) along with modified membership functions “more or less small”, “nor very small” and “rather big” obtained by applying the hedges in Table 2.6. In this table, A stands for

linguistic hedge	operation	linguistic hedge	operation
very A	μ_A^2	more or less A	$\sqrt{\mu_A}$
not very A	$1-\mu_A^2$	rather A	$\text{int}\,(\mu_A)$

the fuzzy sets and “int” denotes the contrast intensification operator given by:

$$\text{int}\,(\mu_A) = \begin{cases} 2\mu_A^2, & \mu_A \le 0.5 \\ 1 - 2(1-\mu_A)^2 & \text{otherwise.} \end{cases}$$

□

2.5 Fuzzy Relations

A fuzzy relation is a fuzzy set in the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$. The membership grades represent the degree of association (correlation) among the elements of the different domains X_i.

Definition 2.16 (Fuzzy Relation) *An n-ary fuzzy relation is a mapping*

$$R\colon X_1 \times X_2 \times \cdots \times X_n \to [0,1], \tag{2.42}$$

which assigns membership grades to all n-tuples $(x_1, x_2, \ldots, x_n)$ from the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$.